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Fractional order Like-Parabolic and Like-Hyperbolic by He's method

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ABSTRACT

This paper talks about Fractional order partial differential equations that considered as one of the recent subjects although the concepts of fractional calculus are old as the beginning of classical calculus with Newton's and Leibentz. The importance of fractional calculus which proved almost all partial differential equations with fractional order and given the best description to problems in different science. Also, the fractional calculus (derivatives and integral) definitions will be given in this paper as the basis which is using to solve problems. The paper in general considers some of models of the main categories where fractional order Like-Parabolic and Like-hyperbolic Partial differential Equations (FPDE's) in one-dimensional will be solved by using He's numerical method. The He's method has accurate and easily to solve many problems. The flexibility and ability of the suggested method to solve different fractional order partial differential problems is illustrated in the present work. He's numerical method has importance in Fractional Calculus, because, it gives a semi analytic solutions for the both linear and nonlinear partial differential equations without more conditions. The using of He's method doesn't need presence of the small-parameters in the PDE's; also, it doesn't need the nonlinearity for the dependent variable and its derivatives. The mathematical researchers, sometime, called this method "a modification of the general Lagrange multiplier method". He's puts this method as a simple and general method to providing an approximation solution to different fractional order differential equations as an iterative sequence to find the final solutions. Different examples will be solved to show the powerful of He's numerical method. Python programs used to solve problems and showing the different between analysis and approximation solutions at different fractional orders. Finally the conclusion of results will be given.

Keywords: Approximation solution for FPDE's, He's numerical method, Like-Parabolic and Like-hyperbolic equations, numerical method for FPDE's, solving fractional PDE's

INTRODUCTION

The Gaussian Difference Continuous Distribution (GDCD) represents a statistical model which illustrates the distinction between two independent Gaussian-distributed variables, which is a very important matter when you are in the error propagation, uncertainty modeling, and step variation in navigation solving. Traditional numerical methods, such as Newton's method, Runge-Kutta solvers, finite difference approximations, Monte Carlo simulations, and differential equation solvers, often assume normally distributed errors. Nevertheless, computational errors in the real world are very often caused by two independent sources of variability interacting, which makes the Gaussian Difference Distribution an actual and truthful representation of numerical uncertainties.

The differential equations (DE's) are meaning, there are changing in the natural and the scientific phenomenon, that the phenomenon are in the over this world, so that these kinds of equations and it solutions, have importance to understand the laws of our world. To obtain the solutions of these equations, there are many Kinds of ways, one from these kinds, ways of mathematics. not all equations have analytic solutions, also some of their didn't have easy method to find the analytic solution, so that we need the other ways to know something about these equations, in these cases we use the Numerical method to find the approximations solutions. The Fractional order Differential equations are generalizing case of the integer order that have the same thing, moreover, Many researchers solve and discussed different fractional differential equations, other researchers used and discussed different methods to solve fractional differential equations analytically and numerically, in [1] using Kudryashov method to solve nonlinear space–time fractional partial differential equations of Burgers type, in [2] using generalized Kudryashov method to solve nonlinear space–time fractional partial differential equations of Burgers type, in [3] using the modified trail equation for non linear fractional differential equations, in [4, 5] using local fractional Sumudu decomposition method for linear fractional partial

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differential equations, in [6-10] using more numerical methods to solve different fractional differential equations, other researchers discussed the system of fractional differential equation, in [11-13] using different numerical methods to solve different systems of fractional differential equations, in [14-16] solving integro-differential equations Volterra or Fredholm with fractional order of derivatives, also, in [17, 18] many researchers generalized kinds of differential equations in sciences (fuzzy, chemical) to fractional order and they used different methods to solve it, in [19] using numerical approach to solve a class of distributed order time fractional partial differential equations, in [20, 21] using Numerical methods and analysis for a multi-term time–space variable-order fractional advection diffusion equations and applications, in [22] the approximation Technique for Fractional Order Delay Differential Equations, in [23] discussed The generalized fractional partial differential equations. In this paper using He's numerical methods to solve certain fractional partial differential equations. The second order has the following general formula

$$A(x,y)\left(\frac{\partial^\alpha}{\partial x^\alpha}\emptyset(x,y)\right)+B(x,y)\left(\frac{\partial^\sigma}{\partial y\partial x}\emptyset(x,y)\right)+\ C(x,y)\left(\frac{\partial^\beta}{\partial y^\beta}\emptyset(x,y)\right)=D\left(x,y,\emptyset,\frac{\partial}{\partial x}\emptyset,\frac{\partial}{\partial y}\emptyset,\dots\right) \tag{1}$$

where $\alpha,\beta\ and\ \sigma=2$

i) If $(B^2-4AC)>0$, then equation has one real characteristic.
ii) If $(B^2-4AC)=0$, then equation has no real characteristic.
iii) If $(B^2-4AC)<0$, then equation has no real characteristic.

we called the these equations are, case (i) hyperbolic equations, case (ii) parabolic equations and case (iii) Elliptic equations, the general applied examples for these cases are

a) If $B=\ D=0$ then equations will be Laplace PDE
$\left(\frac{\partial^\alpha}{\partial x^\alpha}\emptyset\right)+\left(\frac{\partial^\beta}{\partial y^\beta}\emptyset\right)=0;\ \alpha\ and\ \beta=2.$ If $\le\alpha\ and\ \beta\ \le 2$, equation is Like-Laplace FPDE.
b) If B = C = 0 then equation will be One-dimensional, steady-state heat PDE,
$\left(\frac{\partial^\alpha}{\partial t^\alpha}\emptyset\right)=k\left(\frac{\partial^\beta}{\partial x^\beta}\emptyset\right)$, where $\alpha=1 and\ \beta=2$ and k is constant. If $0\le\alpha\le1\le\ \beta\ \le2$,
then equation is Like-heat FPDE.
c) If B = D = 0, then equations will be one dimension wave PDE, $\left(\frac{\partial^\alpha}{\partial t^\alpha}\emptyset\right)=k\left(\frac{\partial^\beta}{\partial x^\beta}\emptyset\right)$where $\alpha=\ \beta=2$ and k is constant, If $\alpha\ >1$ and $\beta\ \le2$ then equation is Like-wave equation FPDE.

NOTE: In the two cases (b and c) equations may be given for two or three dimension, then they have the following forms respectively

$$\frac{\partial^\alpha}{\partial t^\alpha}\emptyset(x,y,t)=k\frac{\partial^\beta}{\partial x^\beta}\emptyset(x,y,t)+L\frac{\partial^\gamma}{\partial y^\gamma}\emptyset(x,y,t) \tag{2}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}\emptyset(x,y,z,t)=k\frac{\partial^\beta}{\partial x^\beta}\emptyset(x,y,z,t)+L\frac{\partial^\gamma}{\partial y^\gamma}\emptyset(x,y,z,t)+R\frac{\partial^\delta}{\partial z^\delta}\emptyset(x,y,z,t) \tag{3}$$

where $(\alpha=1);(\gamma;\delta\ \&\ \beta=2)$ and $R,L,k\ are\ constant$
Also, the (two or three dimensions) like-heat and like-wave fractional differential equations given by taking fractional derivatives, where $0<\alpha\le1<\gamma,\ \delta,\ \beta\le2$.
This paper aligns with the growing body of research in fractional calculus, where fractional-order partial differential equations (FPDEs) are a popular topic of study due to their ability to describe complex phenomena in various scientific fields. Similar to prior works, the paper employs numerical methods to solve FPDEs, a standard approach in computational mathematics. Using the Python tool to solve such a problem, in fact, is in line with the other researches carried out in the past, as its affiliations with scientific communities made it widely used. Such commonalities refer to the fact that this research is an important step in the literature of fractional calculus and the applications of numerical methods.
What makes this paper different is that it deals with fractional-order Like-Parabolic and Like-Hyperbolic equations, a subject that is not usually addressed in the fractional calculus literature. Even though fractional PDEs are frequently studied in more general terms, the paper introduces He's method to solve such differential equations in a particular category, hence, giving an innovative way to solve these problems. One of the main advantages of this paper is that it is based on the method of He which does not confine the use to hitherto small values. This makes it dissimilar to other conventional methods that often depend on such conventions. Besides, He's method applies semi-analytic solutions for the case of both linear and non-linear equations and the existence of this feature as a common one among all synonymous numerical methods enriches the domain and thus the usage of the fractional calculus methods.
In the future, this work has a few possible scenarios that may boost its effectiveness. One direction natural outgrowth of the use of He's method is the one that leads to the handling of not only linear but also nonlinear fractional PDEs with more complex boundary conditions. On the other hand, an advancement in the fields would be to have the same theory applied to the second order equations in several dimensions. This would be one aspect that He's method would grow more practical if it is adapted for multiphysics simulations, which are found typically with agricultural and environmental applications.
It should also be said that the ongoing and previous research successfully showed the versatility of He's method to the one-dimensional fractional PDE, whereas, this might be limiting to the fractional equations of higher orders and/or of the multidimensional system. Indeed, a future task would be to consider the enhancement of the method to deal with diagonally and

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transversally coupled two-dimensional and three-dimensional problems which usually map to the real world. Besides, a better treatment of the boundary conditions of fractional PDEs carry the potential to make the method more robust. Indeed, the computational scalability of the method is a major factor and is the critical factor of a successful application but also in case private use is intended.

One of the weaknesses of the current literature is the lack of an intensive evaluation of the He's method in comparison to the other established fractional calculus methods. Nevertheless, it should be made clear that while He's method has been used in many different problems efficiently and flexibly, a study of methods, still, is of high importance. There are a number of fractional-order differential equations (FOLEs) with other memory kernels, for example, which could be applied to structural vibration analysis. The integration of the theory into reality might be one of the formerly mentioned factors, that allows the theory to convince critical stakeholders of its place in the scientific world.

A considerable impediment to fractional-order PDEs is that they are inherently complex and thus pose a big problem when it comes to their computational study. The methods usually depend on conventional approaches for example the smallness assumptions, the absence of nonlinearity, and the existence of specific boundary conditions, which project their application to some classes of equations. The issue dealt with in this work is to provide a more general and efficient solution without constraints such as the explicit assumption that boundary conditions are given for the problem, fractional-order Like-Parabolic and Like-Hyperbolic equations, according to He's numerical method. The approach outlined here addresses these issues by removing these constraints and rendering a flexible, and accurate solution to the two kinds of equations.

The main task which we set for ourselves in this paper is to discuss and illustrate with examples the utility of He's numerical method in solving such equations. The main goal of this study is to show how accurate, flexible, and effective the numerical method developed by He is, and how it can be used to solve many math problems related to fractional calculus in science. Indeed, the method is practically useful in different areas of science that require solutions based on Nonlinear dynamical systems where vibrational analysis is critical.

BASIC DEFINITIONS [24-27]

RIEMANN FRACTIONAL INTEGRAL (${}_0^R I_x^\theta$) OF ORDER θ

The Riemann Fractional Integral (denoted as ${}_0^R I_x^\theta$)) is a generalization of the classical integral to integral orders, the tool for analyzing functions (including, in particular, fractional orders) of non-integral behavior over their domain. The fractional order θ represents the degree of the integral, and it allows the integration process to be extended beyond the usual integers (e.g., $\theta = 1$ corresponds to a classic integral) [28].

The Riemann fractional integral of a function $F(x)$ of order θ, denoted by ${}_0^R I_x^\theta F(x)$, is defined as:

$${}_0^R I_x^\theta F(x) = \{\frac{1}{\Gamma(\theta)} \int_0^x (x-s)^{\theta-1} F(s) ds,\ \theta > 0\ F(x), \theta = 0 \quad (4)$$

where: $\Gamma(\theta)$ represents the Gamma function, which is just the factorial function generalization for real and complex numbers as well as the integral that includes $(x-s)^{\theta-1}$ making the integral weighted by the function depending on both θ and a distance variable relating x and s.

Above-mentioned equation shows the Riemann fractional integral of order θ can be mathematically be given as the integral of the function $F(s)$ weighted by $(x-s)^{\theta-1}$, where s goes from 0 to x. The factor $(x-s)^{\theta-1}$ introduces a memory effect, meaning the value of the integral at a point x depends on all values of the function $F(s)$ in the range $[0, x]$, but with a weight that decays as s moves away from x.

For $\theta = 0$, the equation simplifies to $F(x)$, as indicated in the second part of equation (4), since the fractional order of the integral becomes zero.

Equation (4) expresses a composition property of the Riemann fractional integrals

$$I_x^\theta\ I_x^\sigma F(x) = I_x^\sigma\ I_x^\theta F(x) = I_x^{\theta+\sigma} F(x) \text{ where } \sigma \text{ and } \theta \geq 0 \quad (5)$$

This relation shows that applying a fractional integral of order σ followed by a fractional integral of order θ is equivalent to directly applying a fractional integral of order $\theta + \sigma$. This composition property highlights the additive nature of fractional integration, where the order of the integration can be combined into a single fractional order.

Thus, the fractional integral operator is closed under composition, meaning that successive fractional integrations can be treated as a single fractional integration of the sum of the individual orders.

1. **Riemann Fractional Integral (Equation 2)**: We generalize the traditional integral to the fractional orders in this article. The integral combines a function $F(s)$ with the weight $(x-s)^{\theta-1}$ where the fractional order θ depends on the weight. This integrator is just the one which can describe the functions whose exponents do not become integers [29].
2. **Composition Property (Equation 3)**: A composition property exists among the fractional integrals, which indicates that the order of integration can be summed up. Namely, applying fractional integrals in a sequence (with orders σ and θ) is the same as a single fractional integral of order $\theta + \sigma$. These two equations build the base for comprehending and practicing fractional integrals in mathematical models that demand non-integer integration, like in fractional calculus and its applications in physical, engineering, and economic systems [30].

RIEMANN FRACTIONAL DERIVATIVES (${}_0^R D_x^\beta$) OF ORDER β

One of the examples of the fractional derivative by operator Riemann ${}_0^R D_x^\beta$ which is a generalization of the classical derivative to fractional orders is an extension to the study of the differentiability of a class of functions that behave like a non-integer. The level

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of the derivative from one point to another in the order β, and the means of fractional derivative are a useful tool in dealing with systems with memory effects a fractional dynamics among others [31].

Riemann's fractional derivative of order β, which is denoted as ${}_0^R D_x^\beta F(x)$, is defined as

$${}_0^R D_x^\beta F(x) = \frac{d^M}{dx^M}\left[\frac{1}{\Gamma(M-\beta)}\int_0^x (x-s)^{(M-\beta)-1} F(s)ds\right] \quad (6)$$

where $M-1<\beta<M$

This equation represents the fractional derivative in terms of a standard integer-order derivative and a fractional integral. Here's how it is structured:

a) **Gamma Function**: The presence of the Gamma function $\Gamma(M-\beta)$ helps normalize the fractional derivative. The Gamma function generalizes the factorial to real and complex numbers and adjusts for the fractional order of differentiation [32].

b) **Fractional Integral**: The integral term $\int_0^x (x-s)^{(M-\beta)-1} F(S)ds$ represents a fractional integral of order $M-\beta$. The function $F(s)$ is integrated with a weigh $(x-s)^{(M-\beta)-1}$, similar to the fractional integral, but with a modification due to the fractional order β [33].

c) **Integer Derivative**: The term $\frac{d^M}{dx^M}$ indicates that we first apply an integer-order derivative M to the result of the fractional integral. This step allows the fractional derivative to be written as an integer-order derivative of a function that itself has a fractional order of integration.

The definition represents the bilateral nature of the relationship between systems showing behavior with powers of a nonintegral order that are more accurate than what is simply an addition of the entire order that exists in the system. Where the first operation, it is a fractional integral of order $M-\beta$, represents the memory effects in the form of non-integer behavior which are separate and distinguishable from the system. This fractional integral results in a non-local cause temporally or spatially, that is, it extends into the past states, which does not mean that the present value is not involved in the process. Fractional systems are more likely to be nonlinear than nonfractional systems in a nonlinear environment. The second operation is an integer-order derivative $\frac{d^M}{dx^M}$ that is used in the fractional integral. Instead, this derivative is a common, integer-order differentiation, the response of the system at a certain point in time or space. Joining together the aforementioned two operations, the equation can account for various system properties including both memory (through the fractional integral) and standard differential behavior (through the integer-order derivative), hence is a useful tool for modeling of real-world events with fractional dynamics [34].

THE COMPOSITION AND DEFINITION

The formula also states that the fractional derivative ${}_0^R D_x^\beta F(x)$ can be expressed as:

$${}_0^R D_x^\beta F(x) = D_x^M I_x^{M-\beta} F(x) \quad (7)$$

This form establishes the relationship between fractional derivatives and fractional integrals. Specifically [35]:

- The fractional derivative ${}_0^R D_x^\beta$ can be viewed as an integer derivative D_x^M applied to a fractional integral $I_x^{M-\beta}$, where M is an integer such that $M-1<\beta<M$.
- It highlights the link between fractional calculus and classical calculus by expressing fractional derivatives as combinations of standard derivatives and integrals.

The constraints concerning terms M and β are mandatory to ensure the validity of fractional operations. In particular, the ratio β must be an integer situated in the interval $M-1\ldots M$, where $M = integer$. This restriction guarantees that the fractional order is suitably limited to avoid the possibility of the latter being overwhelmingly large and thereby the fractional operations would be invalid. Among the applications of fractional derivatives, there are certainly some of them that are not adequately managed by well-known integer-order derivatives and these cases can be anomalous diffusion or the memory effects phenomenon in systems.

Riemann Fractional Derivative (Equation 4) is a function that expresses a concentrated particle's behavior in terms of fractional derivatives of the particles. It is a small step to define the fractional derivatives of a whole number and a fractional unit. This definition enables differential equations for fractional derivatives to be solved using the transformation of the anti-derivative to a fractional derivative and then an algebraic manipulation to convert some algebraic terms to fractional derivatives. Not only does this method give symmetry to Einstein's four space-time dimensions by including them as the fifth and sixth dimensions to the chiral isoplanatic space, it also predicts a generomal-intergran space dominated, non-line space with fractal dimensions. The applications of fractional derivatives and the formula for them in such fields as physics, engineering, and finance which cannot be properly modelled by traditional integer-order derivatives because they involve complex and/or non-local and memory-dependent dynamics make them a flexible and multipurpose tool for researchers and practitioners.

CAPUTO FRACTIONAL DERIVATIVES (${}_0^C D_x^\beta$) OF ORDER β

The fractional derivative Caputo is another extension of the classical derivative to fractional orders. It is often used in physical and engineering applications because it contains a lower-order integer derivative (instead of a higher-order derivative, as in the Riemann definition). The Caputo derivative is more natural for initial value problems. It includes initial conditions of the function and its integer-order derivatives. The Caputo fractional derivative of order β, written as ${}_0^C D_x^\beta F(x)$, is composed in two parts, depending on whether β is fractional or integer [36].

FOR FRACTIONAL ORDERS $M-1<\beta<M$, WHERE M IS AN INTEGER

$${}_0^C D_x^\beta F(x) = \{\frac{1}{\Gamma(M-\beta)}\left[\int_0^x (x-s)^{(M-\beta)-1} \{\frac{d^M}{ds^M} F(s)\}ds\right]; M-1<\beta<M \ \frac{d^M}{dx^M}F(x), \beta = M \quad (8)$$

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The fractional derivative is determined by doing first the $M-th$ derivative of the function $F(s)$, and then, by integrating the result with a weight factor of $(x-s)^{(M-\beta)-1}$. Such a mode of implementation resembles the so-called fractional integral where the integration is carried out from 0 to x, and the weight factor acts as a kind of attenuation function, which adjusts the contribution of each point in the interval. The diminish has the property of slowing down gradually as the distance from x is increasing making it stand out that the fractional derivative is a non-local phenomenon. In this connection, the slope at some point x being the function thereof is not just a unique value — it is more than that — it is the values of the function over the stretch of the points from 0 to x with the most relevant part being the ones closer to x. The more distant ones have a lesser influence.
The $\Gamma(M-\beta)$ term assures the fractional derivative to be properly normalized. It is responsible for the fractional order β of the derivative, by which the scale of the outcome is customized to comply with the concept of this operation being fractional. This normalization is a key factor for the reason that it is the only way to successfully model a disordered fractional system. As a result, it enables the fractional derivative to capture complex, memory-dependent and not necessarily local phenomena that are not included in traditional, integer-order derivatives [37].

FOR INTEGER ORDERS $\beta = M$

$$ {}_0^C D_x^\beta F(x) = \frac{d^M}{dx^M} F(X) \tag{9}$$

- When $\beta = M$, the fractional derivative reduces to the classical integer-order derivative of the function, $\frac{d^M}{dx^M} F(X)$.

The formula for the **Caputo fractional derivative** provides a way to express the derivative as a combination of an integer-order derivative and a fractional integral. Specifically, it states that for $M-1<\beta<M$, the Caputo fractional derivative can be represented as:

$$ {}_0^C D_x^\beta F(x) = I_x^{M-\beta}\left({}_0^C D_s^M F(x)\right) \tag{10}$$

This formula displays that the Caputo fractional derivative of order β is also the fractional integral of order $(M-\beta)$ applied to the $M-th$ derivative of the function $F(x)$. In a nutshell, the implementation of the Caputo fractional derivative when using a differentiating at a fractional order, consists in the first instance of calculating the integer-order $M-th$ derivative of the function and then applying a fractional integral to this result. The $M-th$ derivative of the function and then applying a fractional integral to this result is the fractional integral of order $M-\beta$ what incorporates memory in the formulation or non-local behavior which is the nature of fractional calculus. This means the latter is a problem where the derivative at a point leads to the entire interval until this point weighted by the fractional order. This form of the derivative offers users an easier time with the concept and computation of fractional derivatives which in turn is then applied to the modeling of phenomena with fractional dynamics or memory effects [38].

KEY POINTS

a) Caputo Fractional Derivative: As for the history of turning ideas into math, Caputo fractional derivative appears to be the updating of the traditional derivative. In the situation of fractional orders $M-1<\beta<M$, this derivative is a combination of the $M-th$ derivative of the function and the integration with a fractional kernel $(x-s)^{(M-\beta)-1}$. On the other hand, the Riemann fractional derivative is based on a more or less direct combination of integer derivatives and fractional integrals.
b) Normalization and Integer Derivative: The $\Gamma(M-\beta)$ term within the action formula ensures the fractional order β is properly normalized. In the other situation when $\beta = M$, the fractional derivative is reduced to a standard integer-order derivative.
c) Fractional Integral-derivative Relationship: The formula highlights the close connection between fractional derivatives and fractional integrals. The novelty here is that when considering the Caputo fractional derivative, one can think of applying the fractional integral to the integer-order derivative of the function.
d) Applications: One of the major advantages of the Caputo fractional derivative is that it provides the data concerning the initial condition of the function and its integer acquisitions, those being viscoelasticity, or other systems with fractional dynamics [38].

- Caputo Fractional Derivative (Equation 5) is a distinguishing character of the fractional derivatives that extend the classical derivatives to the fractional orders as it involves both an integer derivative and a fractional integral.

- The fractional derivative is given as the $M-th$ derivative of the function, whereby the superposition or desuperposition by a multiplying factor that is a function of β is done.

- β=M corresponds only to the standard integer derivative case.

- This conception is of utmost importance in practical problems that can simulate situations where both the function and its derivatives will change at the same time but initial conditions for both of them are already known, e.g., in physics.

JUMARIE FRACTIONAL DERIVATIVES $\left({}_0^J D_x^\beta\right)$

The **Jumarie Fractional Derivative** is a definition of fractional derivatives based on a generalization of the classical difference operator. It provides a way to extend the concept of differentiation to non-integer (fractional) orders. This definition is useful for

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modeling systems that exhibit fractional behaviors or anomalous dynamics, where traditional integer-order derivatives might not be sufficient [39].
The Jumarie fractional derivative of order β, denoted ${}_0^J D_x^\beta F(x,t)$, is defined as:

$${}_0^J D_x^\beta F(x,t) = \frac{\Delta^\beta (F(x,t) - F(x,0))}{h^\beta} \quad (11)$$

where: $F(x,t)$ is the function to be differentiated with respect to x (and possibly time t), Δ^β represents the fractional difference operator, which generalizes the concept of the finite difference and h is the small step size, and as $h \to 0$, it approaches the limiting value of the fractional derivative.

KEY POINTS IN THE FORMULA

The Jumarie fractional derivative formula is a critical component of the concepts that extend the classical concept of differentiation to the fractional orders. The fractional difference operator Δ^β is a new form of the finite difference operator and is used in numerical methods to approximate derivatives. The non-integer derivative of the first order is a differential operator, with the non-integer power of the step size h that acts as a scaling factor to the operator making it the is the fractal one on the systems whose memory behavior the operator captures both fractional dynamics and memory-dependent behavior of the system.
The fractional derivative is obtained in the limiting process as $h \to 0$, and it is similar to classical derivatives but instead of having integer powers, we have fractional powers of h. This fractional scaling is much desirable when it comes to coping with systems that manifest memory effects or fractional behavior, e.g. (the so-called) anomalous diffusion or other phenomena of this kind that have long-range dependencies.
A noteworthy property of the Jumarie fractional derivative is that it passes from non-fractional forms to standard integer-order derivatives for large values of β. In this way, the fractional derivative becomes a classical integer-order derivative in case β is an integer which paves the way for continuity among the fractional and classical differentiation.
The derivative of a constant function $F(x,t) = c$ is the zero constant, no matter what the value of β is (if $0 < \beta < 1$), which is in line with the derivative of a constant function in classical calculus. Thus, we can know the Jumarie fractional derivative is properly set up in simpler cases as well.
The general definition of the Jumarie fractional derivative is done by using the fractional difference operator Δ^β as the fractional order, i.e., β. This makes it a very flexible method for fractional differentiation because it can include the integer-order derivatives too. The method's ability to include integer-order derivatives as a special case when β is an integer ensures that the approach smoothly bridges classical and fractional calculus.
The derivative's limit process, which contains fractional powers of the step size h, stands distinct due to the non-integer behavior thereof. This is crucial for applications involving fractal dynamics, anomalous diffusion, and systems with memory or complex scaling. These features make the Jumarie fractional derivative a powerful tool for studying systems with fractional dynamics.

${}_0^J D_x^\beta$ OF COMPOUNDED FUNCTIONS

Jumarie fractional derivative μ fractional integral are in fact derivatives and integrals over one standard that is only applicable to the entire degree. They are used in the models of non-integer-degree or non-integer behavior that is not suitable for classical integer-order calculus. Highlight here by using the Jumarie fractional derivatives and integrals that they are a way to attend to a data set of a really fractional order function and therefore work such as for $0 < \beta < 1$ [40].
The fractional derivative of a compounded function $F(x,t)$ is however given by:

$${}_0^J D_x^\beta F(x,t) = \Gamma(1+\beta) \frac{dF(x,t)}{dx^3} \quad (12)$$

On the other hand, the certain $\Gamma(1+\beta)$, the Gamma function, is the extended or generalized idea of factoral function to the non-integer numbers of the factorial function which in practical is used for the normalization of the fractional derivative and that, τ the order of the fractional derivative, where $0 < \beta < 1$ is.
The formula given above says that the Jumarie fractional derivative of a function corresponds to the fractional power derivative with a scale factor is the Gamma function. This definition makes differentiation possible for non-integer orders and thus allowing the emergence of a new area for the treatment of phenomena which require fractional differentiation.

FRACTIONAL INTEGRAL (JUMARIE INTEGRAL)

The fractional integral with respect to t, also known as the Jumarie integral, is given as:

$$F(x,t) = \int_0^t F(x,s)(ds)^\beta = \beta \int_0^t F(x,s)(t-s)^{\beta-1} ds \quad (13)$$

where: A fractional order β is included in the integral, so $0 < \beta < 1$, and the integrand is multiplied by $(t-s)^{\beta-1}$, producing a memory effect—$F(x,s)$ from earlier times s influence the value $F(x,t)$ with a fading weight that occurs as $(t-s)$ becomes larger in magnitude.
This fractional integral is a generalization of the traditional integral, where the order of integration is not necessarily an integer. It describes processes where the history of the function influences its future evolution, and the impact of past events decays with time, captured by the fractional power β [41].

SOLUTION TO FRACTIONAL DIFFERENTIAL EQUATIONS

In the context of fractional differential equations, if $F(x,t)$ is a solution to a differential equation, then it can be expressed as a fractional integral. Specifically [42]:

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$F(x,t) = \int_0^t F(x,s)\,(ds)^\beta$ (14)

The integral respect to $(dt)^\beta$, given by jumarie as the solution of fractional differential equations as
$d\,F(x,t) \cong F(\mathrm{x},\mathrm{t})\,(dt)^\beta$, where $t \geq 0$, $0 < \beta < 1$ and $F(x,0) = 0$, For a continuous function
This is a general form of the solution to a fractional differential equation, where the function $F(x,t)$ depends on its history, and the past values of $F(x,t)$ are weighted by $(t-s)^{\beta-1}$.
For example, if $F(x,t) = t^k$, we can compute the fractional integral:
$F(x,t)$ is a solution of differential equation, for $(0 < \beta < 1)$
define $F(x,t)$ as $F(x,t) = \int_0^t F(x,s)\;(ds)^\beta = \beta\int_0^t F(x,s)\;(t-s)^{\beta-1}ds$
Let $F(x,,t) = t^k$ then $\int_0^t s^k\;(ds)^\beta = \frac{\Gamma(1+\beta)\Gamma(1+k)}{\Gamma(k+1+\beta)}\,t^{k+\beta}$ (15)
This result shows how the Jumarie fractional integral affects a power function t^k, providing an expression for the solution to a fractional differential equation when the function is a power of t.

a) **Jumarie Fractional Derivative**: The Jumarie fractional derivative of order β is a broader definition of the classical derivative incorporating a scaling factor $\Gamma(1+\beta)$; thus, we can differentiate functions that are endowed with a new property called the order

b) **Jumarie Fractional Integral**: The Jumarie fractional integral is a generalization of the ordinary integral. It consists of the function weighted by $(t-s)^{\beta-1}$ and this aspect incarnates the memory effect in the system so that past values of the function also contribute to the current value.

c) **Fractional Differential Equations**: The solutions of the differential equations, in fractional terms, can give rise to fractional integrals, which refer to the past of the function with a fractional weight. This approach is suitable for representing systems with memory, anomalous diffusion, or fractional dynamics.

d) **Application**: Fractional derivatives and integrals allow the definition of a very important tool in physics, engineering, biology, and finance, where kinds of things develop into very complex, non-integer value forms that cannot be treated by traditional integer-order calculus [43].

PROPERTIES OF FRACTIONAL DERIVATIVES OF ANALYTIC FUNCTIONS [44]

The section describes a detailed approach for solving fractional partial differential equations using **He's method** and other related techniques such as Jumarie fractional derivatives, as well as various formulations for fractional derivatives of analytic functions. Here's an interpretation and explanation of the key concepts.

FRACTIONAL DERIVATIVES OF ANALYTIC FUNCTIONS

The section begins by detailing some properties of fractional derivatives for specific analytic functions [45].

i) Fractional Derivatives of Powers of x

- $D_x^\beta\,(x^K) = \frac{\Gamma(K+1)}{\Gamma(K-\beta+1)}\,(x^{K-\beta})$ where $0 < \beta \leq K$ (16)

This formula describes how the fractional derivative of a power of x behaves, showing that it results in another power of x, but with a modified exponent due to the fractional order β. The Gamma functions Γ appear to normalize the result, reflecting the non-integer order of the derivative.

- For negative powers of x: $D_x^\beta\,(x^K) = (-1)^\beta\,\frac{\Gamma(K+\beta)}{\Gamma(K)}\,(x)^{-(K+\beta)}$, this property describes the behavior of the fractional derivative of a negative power of x. The result involves a phase factor $(-1)^\beta$ due to the fractional nature of the derivative, as β is non-integer.

- $D_x^\beta\,(x^{-K}) = (-1)^\beta\,\frac{\Gamma(K+\beta)}{\Gamma(K)}\,(x^{-(K+\beta)})$ where $(-1)^\beta = e^{i\beta\pi}$ (17)

i) **Product Rule for Fractional Derivatives**: A product of two functions $f\,(x)$ and $g\,(x)$ is given by the fractional derivative
$D_x^\beta\,[f(x)g(x)] = \{\sum_{j=0}^{\infty}(\beta\;j)\;D_x^{\beta-j}[f(x)]\,D_x^j(g(x))\;\;D_x^\beta[f(x)]C\;\;;if\;\;g(x) = C;$ (18)
where C is constant.
Derived from the product rule in fractional calculus is a generalized form, apparently, such that each of the derivative orders is the subject of the distribution between the two given functions $f(x)$ and $g(x)$.

ii) Fractional Derivative of Exponentials

An exponential derivative function e^x is actually a Taylor series expansion which can be considered as a generalized version of exponential functions. The fractional derivative of the exponential function e^x is equivalent to an infinite series that can be interpreted as a fractional exponential function E_β^x, which is a generalization of the traditional exponential function [46].

iii) Fractional Derivative of e^{ax}

From (iv) $E_{-1}^x = D_x^{-1}\,(e^x) = \sum_{j=0}^{\infty}\frac{x^{j+1}}{\Gamma(j+2)} = E_\beta^x \;= e^x - 1 = \int_0^x e^s\,ds$ (19)
Let a = -1, and Since $((-1)^\beta = e^{i\beta\pi}$), then by using (iv and v) we get
$D_x^\beta\,(e^x) = D_x^\beta\left[\sum_{j=0}^{\infty}\frac{(ax)^j}{j!}\right] = a^\beta\,\sum_{j=0}^{\infty}\frac{(ax)^{j-\beta}}{\Gamma(j-\beta+1)} = a^\beta\,E_\beta^x$ (20)

also, step by step, one can find

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$D_x^{\beta}\ (e^{ax}) = a^{\beta}\ e^{ax}$ (21)
$D_x^{1}\ (e^{ax}) = a^{1}\ e^{ax}$ (22)

$D_x^{2}\ (e^{ax}) = a^{2}\ e^{ax}$ (23)

⋮

$D_x^{n}\ (e^{ax}) = a^{n}\ e^{ax}$ (24)
Now if $n = \beta$ we can rewrite same formula for fractional derivative as $D_x^{\beta}\ (e^{ax}) = a^{\beta}\ e^{ax}$ (25)

iv) Complex Exponentials

If $= \sqrt{-1}$, the fractional derivative takes a form involving complex exponentials
Let $a = \sqrt{-1}$, Since $D_x^{\beta}\ (e^{ax}) = a^{\beta}\ e^{ax}\ and\ i = e^{i\pi/2}$ (26)
one can rewrite fractional derivative as

$D_x^{\beta}\ (e^{ax}) = a^{\beta}\ e^{(i\beta\pi/2)}\ e^{(iax)} = a^{\beta}\ e^{i\left(\frac{\beta\pi}{2}+ax\right)}$ (27)
or $D_x^{\beta}\ (e^{ax}) = a^{\beta}\left(ax+\frac{\beta\pi}{2}\right) + i\,sin\,sin\,\left(ax+\frac{\beta\pi}{2}\right))$ (28)
The fractional derivatives of order β can be written by Trigonometric function as,
or $D_x^{\beta}\ (e^{ax}) = a^{\beta}\left(ax+\frac{\beta\pi}{2}\right) + i\,sin\,sin\,\left(ax+\frac{\beta\pi}{2}\right))$ (29)

$D_x^{\beta}\ [\,sin\,sin\,(ax)\,] = a^{\beta}\ sin\,sin\,\left(ax+\frac{\beta\pi}{2}\right)\ D_x^{\beta}\ [\,sin\,sin\,(ax)\,] = a^{\beta}\ sin\,sin\,\left(ax+\frac{\beta\pi}{2}\right)$ (30)

This turns out a fraction of a derivative be interpreted as a periodic function with a phase $\beta\pi/2$. This is a bogus issue in modeling the periodic phenomena of oscillatory or wave-like ones in systems of fractional dynamics.

HE'S METHOD FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

His approach aims to find solutions of fractional partial differential equations (PDEs) through the iteration of correction functions. To clarify, an algorithm was developed, which was based on steps of solving differential equations in a simple way, and we can write the one-dimension and non-linear Fractional partial differential equation, and by using the operators' form as shown in the second expression here. At the same time, fractionation indicate the addition and the opposite for waves at differ [47].

$$L\emptyset(x,t) + K\emptyset(x,t) = f(x,t) \tag{31}$$

where, $L = \frac{\partial^{\alpha}\emptyset(x,t)}{\partial t^{\alpha}}$ is a linear differential operator, $0 < \alpha \le 2$, $k = \frac{\partial^{\beta}\emptyset(x,t)}{\partial x^{\beta}}$ is a nonlinear differential operator, $0 < \beta \le 2$, $f(x,t)$ is a analytical function (a sores function). The solution of this equation by using He's method, given by the general steps

STEP-BY-STEP ALGORITHM FOR HE'S METHOD

- **Step 1**: The general solution to the fractional PDE (equation 6) is given by correctional functions $\emptyset_{n+1}(x,t)$, which are iteratively updated from the previous approximation $\emptyset_{n+1}(x,t)$. This correction is governed by the fractional differential operators and the Lagrange multiplier λ:

$$\emptyset_{n+1}(x,t) = \{\emptyset_n(x,t) + \int_0^t \lambda(x,s)\,(L\,\emptyset_n(x,s) + K(\emptyset_n)(x,s) - f(x,s)\}ds\};\ \alpha = 1,2 \tag{31}$$

or by equivalent Jumarie's fractional definitions, $1 < \alpha < 2$

$$\emptyset_{n+1}(x,t) = \left\{ \begin{array}{c} \emptyset_n(x,t) + \int_0^t \lambda(x,s)(L\,\emptyset_n(x,s) + K(\emptyset_n)(x,s) - f(x,s)\}ds\,;\ \alpha = 1,2 \\ or\ by\ equivalent\ \text{Jumarie's Fractional definitions};\ 1 < \alpha < 2 \\ \emptyset_n(x,t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(x,s)\{L\,\emptyset_n(x,s) + K\,\widetilde{\emptyset_n}(x,s) - f(x,s)\}(ds)^{\alpha} \end{array} \right\} \tag{32}$$

where L and K represent linear and nonlinear differential operators respectively, and $f(x,t)$ is an analytical function, and where, λ is general Lagrange's multiplier, must find it, $(\emptyset_n)$ means getting stationary function and $\emptyset_{n+1}(x,t)\,, n \ge 0$ are approximation solutions that we can find it after selecting the first approximation solution $(\emptyset_0)$, by using the conditions that given with the problems (i.e. the initial conditions, boundary conditions or both).

- **Step 2**: The variation operator δ is applied to both sides of the correction equation to derive the necessary conditions for the Lagrange multiplier. These conditions help to obtain an expression for the Lagrange multiplier λ.

Take (δ) to both sides of Eq. 7 where $\delta(\emptyset_n) = 0$, we get

$$\delta\emptyset_{n+1}(x,t) = \delta\emptyset_n(x,t) + \delta\int_0^t \lambda(x,s)\{L\,\emptyset_n(x,s) + K\,\widetilde{\emptyset_n}(x,s) - f(x,s)\}ds \tag{33}$$

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- **Step 3**: The Lagrange multiplier is computed using integral techniques (e.g., integration by parts), ensuring that the correction to the solution is stationary with respect to changes in the function $\emptyset_n(x,t)$. This provides an update rule for the solution.

Find Lagrange's multiplier (λ) by using integral by parts to solve Eq. 8 with helping the stationary condition ($\delta(\widetilde{\emptyset_n}) = 0$), Then we will get the following notes

$L = \frac{\partial}{\partial t}$, *with* $\delta(\emptyset_n) = 0$ *also* $\delta\, K(\emptyset_n) = 0$, then we have

$\lambda' = 0 \qquad (\lambda + 1)|_{s=t} = 0$ (34)

So that $\lambda = -1$, put it in Eq. 8, we get

$\emptyset_{n+1}(x,t) = \emptyset_n - \int_0^t \{L\, \emptyset_n(x,s) + K\, \widetilde{\emptyset_n}(x,s) - f(x,s)\}\, ds$ (35)

i) If $L = C\frac{\partial^2}{\partial t^2}$, where C is constant, then by same way we get

$\delta\emptyset_n + C\,\lambda(s)\ \delta\emptyset_n{}'(s)|_{s=t} - C\lambda'(s)\delta\emptyset_n(s)|_{s=t} + \int_0^t C\lambda''(s)\delta\emptyset_n(s)ds = 0$ (36)

Equation gives the stationary conditions as

$C\lambda'' = 0\ 1 - C\lambda'(s)|_{s=t}\ \lambda(s)|_{s=t} = 0$ (37)

solution of these equations gives stationary value of λ,

$\lambda = \frac{1}{C}(s-t)$, put this solution in Eq. 9 yield

$\emptyset_{n+1}(x,t) = \emptyset_n(x,t) + \left(\frac{1}{C}\right)\int_0^t (s-t)\left\{L\, \emptyset_n(x,s) + K\, \widetilde{\emptyset_n}(x,s) - f(x,s)\right\}ds$ (38)

ii) If $L = C\frac{\partial^2}{\partial t^2} + h\frac{\partial}{\partial t}$, where C and h are constants, by the same way we get the stationary conditions and stationary value of (λ) as $\lambda = \left(\frac{1}{C}\right)\left[e^{\left(\frac{h}{C}\right)(s-t)} - 1\right]$ (39)

- **Step 4**: Using initial and boundary conditions, the first approximation $\emptyset_{n+1}(x,t)$ is obtained. For fractional initial value problems, boundary value problems, or initial-boundary value problems, specific formulations of the solution are given, incorporating the conditions.

The differential equations (initial value problems, boundary problems or initial boundary problems), initial conditions only, boundary conditions only can be used or both conditions at same time) can be used, to find the first (zero) approximation solution $\emptyset_0(x,t)$, this property gives three choices as follows

i) For fractional order (α) initial value problems, where ($M-1 \le \alpha \le M$, M is integer and $0 \le x \le 1$, $t > 0$, in this case ($M-1$) initial conditions has been obtained,

that given as, $\frac{\partial^j}{\partial t^j}\emptyset\,(x,t)|_{t=0}; j = 0,1,\dots,M-1$, then the first approximation solution given as

$\emptyset_0(x,t) = \sum_{j=0}^{M-1} t^j\ \frac{\partial^j}{\partial t^j}\emptyset\,(x,t)|_{t=0}$ (40)

ii) For initial-boundary value problems, in these problems; both conditions can be used as

Let $(1 < \alpha\,, \beta \le 2), 0 \le x \le 1, t > 0$, for this order problems have the following conditions

$(I.C)_1 = \emptyset(x,0) = g_0(x),\ (I.C)_2 = \emptyset_t{}'(x,t)|_{t=0} = g_1(x)$ (41)

$(B.C)_1 = \emptyset(0,t) = H_0(x),\ (B.C)_2 = \emptyset(1,t) = H_1(x)$ (42)

The first approximation $\emptyset_0^*(x,t)$ given as

$\emptyset_0^*(x,t) = \emptyset_0(x,t) + (1-x^m)[g_0(x) - H_0(x)] + x^m(g_1(x) - H_1(x))$ (43)

iii) For fractional initial value problems, one can use the following method to choose $\emptyset_0(x,t)$, to get one step faster than the method in formula (11),

Let $(1 < \alpha\,, \beta \le 2), 0 \le x \le 1, t > 0$, a new method given as

$\emptyset_0^*(x,t) = g_0(x) + tg_1(x) + t^2\, g_2(x)$ (44)

where, $g_2(x)$ is the result after putting $\emptyset_0(x,t)$ in problem

- **Step 5**: The process is repeated iteratively (steps 1 to 4), refining the approximation $\emptyset_{n+1}(x,t)$ until it converges to the final solution. The n^{th} approximation converges to the final solution as $n \to \infty$: $\emptyset(x,t) = \emptyset_n(x,t)$ (45)

APPLICATION TO FRACTIONAL DIFFERENTIAL EQUATIONS

Schwartz' method gives a powerful and versatile way to solve nonlinear fractional partial differential equations (FPDEs). By adopting fractional derivatives and integrals, it is a handy tool to solve those systems which have difficult memory twice or exhibit non-local behavior that is quite common in fractional calculus. The method is developed in such a way that a recursive, iterative process is utilized to solve the problem by using the correction function progressively at each step. The Lagrange multiplier is a core part to the correctness of the correction step and the accuracy of the approximation as the method goes on. The technique follows an iterative procedure until the solution converges, and this is enough to present the system's behavior properly. The fractional order differentiation affects the fractional derivatives of analytic functions having powers, exponentials, and trigonometric functions that develop in a manner consistent with that of differentiation by integer order. His technique works best with nonlinear systems in which a step-by-step strategy is employed. It involves fractional operators, correction functions, and

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Lagrange multipliers and utilizes the initial and boundary conditions to steer the process, which eventually produces an accurate solution for complex fractional differential equations [48].

CONVERGENCE OF HE'S METHOD

This section, we are going to take a closer look at the convergence of He's method of solving fractional differential equations. He's approach for solving these equations attempts to find the solution by using iterative correction functions and finding the limit of these iterations, which is assumed to converge to the true solution of the problem. The explanation below gives a detailed version of the method, where the focus is on the method steps, the use of correctional functions, and the utilization of mathematical theorems to ensure convergence is going to be.

FORMULATION OF DIFFERENTIAL EQUATIONS (EQUATION 15)

He's method is applied to solve differential equations of the second order. These equations typically include both spatial and temporal derivatives, as seen in the general equation.

He's method gives the solution of differential equations as, $\emptyset\,(x,t) = \emptyset_n(x,t)$ a, after using the recurrence sequence of functions. So that, one can rewrites this case by the following mathematical formulas, [49]

$$F\left(\phi(x,t), \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial^2\phi}{\partial t^2}, \frac{\partial^2\phi}{\partial x^2}, \frac{\partial^2\phi}{\partial t\partial x}\right) = 0 \quad (46)$$

where, $\phi\,(x,t)$ is the unknown function to be solved, F is a general form of second order partial differential equations, correspond with specified initial conditions.

This equation represents the core problem He's method aims to solve. The function F is generally derived from the physical system being modeled, where $\phi(x,t)$ could represent quantities like temperature, displacement, or concentration in various scientific fields.

CORRECTIONAL FUNCTION APPROACH (EQUATION 16)

He's method proceeds by transforming the differential equation into a recursive correction function in t direction [50]

$$\phi_{n+1}\,(x,t) = \phi_n(x,t) + \int_0^t \lambda(x,s)\, F\left(\tilde{\phi}_n(x,s), \frac{\partial\phi_n}{\partial s}, \frac{\partial\tilde{\phi}_n}{\partial x}, \frac{\partial^2\phi_n}{\partial s^2}, \frac{\partial^2\tilde{\phi}_n}{\partial x^2}, \frac{\partial^2\tilde{\phi}_n}{\partial s\partial x}\right) ds = 0 \quad (47)$$

where: $\phi_n(x,t)$ is the $n-th$ approximation of the solution, and $\tilde{\phi}_n(x,s)$ represents the stationary form of the correction function, and $\lambda(x,s)$ is the Lagrange multiplier, a term used to adjust the corrections to ensure the solution adheres to boundary and initial conditions.

The integral represents the correction applied in the t-direction, integrating over the previous iterations to refine the solution. i.e. this recursive correction step is repeated until the approximation sequence converges to a solution of the differential equation.

$\tilde{\phi}_n$ is considered as He's monographs, (i.e $\delta(\tilde{\phi}_n) = 0$). To find the optimal value of λ.

DEFINING THE FUNCTIONAL AND ITS VARIATION (EQUATION 18)

The correction functional is obtained as stationary by taking the derivative (δ) of both sides. This leads to the result in Equation 16, [51]

$$\delta\phi_{n+1}\,(x,t) = \delta\phi_n(x,t) + \delta\int_0^t \lambda F\left(\tilde{\phi}_n(x,s), \frac{\partial\phi_n}{\partial s}, \frac{\partial\tilde{\phi}_n}{\partial x}, \frac{\partial^2\phi_n}{\partial s^2}, \frac{\partial^2\tilde{\phi}_n}{\partial x^2}, \frac{\partial^2\tilde{\phi}_n}{\partial s\partial x}\right) ds = 0 \quad (48)$$

So, the results will be stationary conditions and consequently the optimal value of λ obtained.

(i.e the solution of the differential equation is considered as the fixed point of the Eq. 16 under the suitable choice of the initial term $\phi_0(x,t)$. by using the method that is using some concepts of variation in calculus and considered in definition, we can see that the variable quantity $v(\phi\,(x,t)$ is a functional dependent on a function $\phi(x,t)$, there are functions $\phi(x,t)$ of a certain class of functions, that correspond a value v.

To formalize the idea of minimizing the correction functional, we define a functional $v(\phi(x,t))$ that depends on the function $\phi(x,t)$. The variation $\delta v\phi(x,t)$ is then computed as

$$\delta v(\phi(x,t)) = \frac{\partial}{\partial s}\, v[\phi(x,t) + s\,\delta\,\phi(x,t)]|_{s=0} \quad (49)$$

This process calculates how the functional changes as we perturb the function $\phi(x,t)$. Setting this variation equal to zero ensures that the correction step produces a valid solution that minimizes the functional, thus leading to an optimal solution.

Theorems for Convergence [52]

The convergence of He's method relies on two theorems: the First Theorem and Banach's Fixed Point Theorem

Theorem 1: If a functional $v(\phi(x,t))$ achieves a maximum or minimum at some point $\phi\,(x,t) = \phi_o(x,t)$, then the variation $\delta\phi_o(x,t) = 0$. This means that the correction is optimal when the variation is zero, which guarantees that the correction step is valid.

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Theorem 2: **Banach's Fixed Point Theorem**: This theorem is a key result that guarantees the existence and uniqueness of a solution to a nonlinear equation. It states that if the mapping G is a contraction (i.e., the distance between two points after applying the mapping is smaller than their original distance by a constant factor), then the sequence of iterates will converge to a unique fixed point. For He's method to converge, we need the functional G to be a contraction mapping, meaning the corrections to the function $\phi\,(x,t)$ will eventually shrink as the iterations proceed.

Assume that X be a Banach space and $G: X \rightarrow X$, is a nonlinear mapping, and suppose that, For some constant ($\zeta < 1$) Then, G will have a unique fixed point, Furthermore, sequence ($\phi_{n+1}(x) = G(\phi_{n+1}(x))$, With an arbitrary choice of $\phi_0(x) \in X$, will converge to the fixed point of G and, According to Theorem two, for the nonlinear mapping, a sufficient condition for convergence of the He's Method will be, (strictly contraction of G), Furthermore, the sequence Eq. 14 converges to the fixed point of G which also is the solution of the differential equations in Eq. 15.

Final Solution

The final solution of the differential equation is obtained as the limit of the sequence $\phi_n(x)$ as $n \rightarrow \infty$: $\phi(x,t) = \phi_n(x,t)$ (50)

He is powerful and the augmented Lagrange Hopfield method for solving the following initial value problem is based correction function succced iterative refinement. In each step, a correcting function, obtained from the differential equation, is altered via the Lagrange multiplier λ for the bettering of the approximation. This recursiveness will be applied until the sought-after solution reaches the tolerance limit. He method guarantees that each correction step passes the test for consistency that is a stationary condition under which the variation of the functional=0. Thus, the solution continuously is accurate and consistent at all the levels of iterations. The He method converges mathematically via Banach's Fixed Point Theorem that states that if the whole iterative process is a contraction mapping, then the approximating sequence absolutely converges to the unique solution to the initial value problem. It can be seen from the fact that the method leads to the final solution which is the limit of the sequence of the approach. In this regard, the approach is characterized by the He method which can be considered as a more appropriate method for solving nonlinear fractional partial differential equations, it provides convergence and a good solution in the case of complicated systems.

RESULTS AND DISCUSSION: NUMERICAL EXAMPLES

EXAMPLE 1

By He's method solve fractional partial differential equation

$$ {}_0^J D_t^{\alpha} \emptyset(x,t) = a\left({}_0^J D_x^{\beta}\right)\emptyset(x,t) + G(x,t) \tag{51} $$

where $a \in R$, $0 < x < 1, t > 0$, $1 < \alpha\,; \beta \leq 2$ and $G(x,t) = a^{\alpha} e^{x} e^{at}\,(1 - a^{1-\alpha})$, and initial conditions $\emptyset(x,0) = e^{x}$, $\emptyset_t'(x,0) = ae^{x}$.

KEY STEPS IN HE'S METHOD

i) Approximation for first the technique

The first approximation $\emptyset_0\,(x,t)$ is chosen based on the initial conditions. This is often the standard approximation given by:

$$ \emptyset_0\,(x,t) = e^{x}(1 + \alpha t) \tag{52} $$

- The correction function is then applied using the Lagrange multiplier $\lambda = (s - t)$. The correction function is given by:

$$ \emptyset_{n+1}(x,t) = \emptyset_n(x,t) + \int_0^t (s-t)\left(D_s^{\alpha}\emptyset_n(x,s) - D_x^{\beta}\emptyset_n(x,s) - G(x,s)\right)ds \tag{53} $$

- This iterative process is repeated for each step until the solution converges.

ii) Second method for approximation

$$ \emptyset_0^*(x,t) = e^{x}(1 + at + a^{\alpha}t^2 + a^{1+\alpha}t^3) \tag{54} $$

- The corrections are then applied using similar steps as in the first method.

iii) Convergence to Exact Solution

- By continuing the iterative process, the approximations $\varphi_n(x,t)$ converge to the true solution. In the case where $\alpha = 2$ and $a = -1$ or $a = 1$, both the second and the first methods yield the exact solution, which involves expansions in terms of t:
- For $a = -1$

$$ \emptyset(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{at^4}{4!} - \frac{at^5}{5!} + \cdots\right) \tag{55} $$

- For $a = 1$

$$ \emptyset(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{at^4}{4!} + \frac{at^5}{5!} + \cdots\right) \tag{56} $$

iv) Error Analysis

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- As the number of terms increases, the approximation becomes more accurate. The error between the exact and numerical solutions decreases as more terms are included in the series expansion.

Error and Approximation Analysis (Table 1)

The results of the approximation for different values of α and β are shown in the table, along with the maximum errors (Max-Err) between the exact solution and the approximation.

Table 1. Error and approximation analysis

Figure	Terms	$0\le x\le1$	$0\le t\le2$	$1\le\alpha\le2$, $\beta=2$	Curves of Exact and Numerical	Max-Err
A	5	X=0: 0.1:1	t=0:0.2: 2	α =1.3, β=2	Curves between t and $\phi(x,t)$	1.0245
B	5	X=0:0.1:1	t=0:0.2: 2	α =1.5, β=2	Curves between t and $\phi(x,t)$	0.6403
C	10	X=0:0.1:1	t=0:0.2: 2	α =1.5, β=2	Curves between t and $\phi(x,t)$	0.4571
D	10	X=0:0.1:1	t=0:0.2: 2	α =1.8, β=2	Curves between t and $\phi(x,t)$	0.1590
E	20	X=0:0.1:1	t=0:0.2: 2	α =2, β=2	Exact solution=Numerical	$6.5015e^{-13}$

From Table 1, we observe that as the number of terms (approximations) increases, the **maximum error** decreases significantly. For $\alpha = 2$ and $\beta = 2$, the error becomes extremely small, approaching the exact solution. The exact solution matches the numerical approximation at a very high precision when many terms are included.

Fig. 1 shows the comparison between the approximation and exact solution for $U(x,t)$, with t on the x-axis and $U(x,t)$ on the y-axis. Each curve represents a different value of x, with both the approximation (dashed lines) and exact solutions (solid lines). As the number of terms in the approximation increases, the curves for the approximation approach the exact solution, demonstrating the accuracy of He's method for solving fractional partial differential equations.

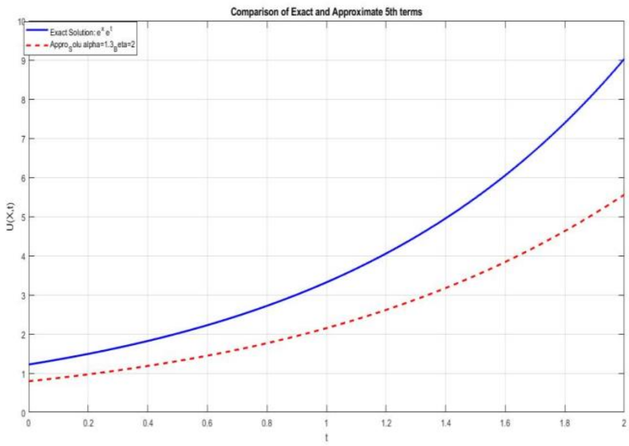

Fig. 1. Comparison of Approximation and Exact Solution for $U(x,t)$ at various x values. The dashed lines represent the approximation of the fractional partial differential equation using He's method, while the solid lines show the exact solution

Fig. 2 presents a comparison between the **approximation** and **exact solution** for the function $U(x,t)$, across the domains $0 \le x \le 10$ and $0 \le t \le 20$. This comparison serves to evaluate the effectiveness of **He's method** in solving fractional partial differential equations. The plot provides insight into how the approximation evolves and approaches the exact solution as the iterative process progresses.

we observe the **approximation** of $U(x,t)$ obtained using He's method. Initially, the approximation shows some deviation from the exact solution, indicating that the method is still refining the solution. The surface gradually becomes smoother as more terms are added to the approximation, with the function becoming closer to the exact solution. This demonstrates the iterative correction process employed by He's method, which improves the accuracy with each step. As more terms are considered, the approximation becomes more accurate, showing the method's ability to progressively approach the true solution.

The exact solution is computed directly and remains constant for the given domain. The smooth surface in this plot reflects the exact, theoretical behavior of $U(x,t)$ across both the spatial and time variables. The exact solution serves as a reference for comparing the approximation, providing a benchmark to evaluate the method's performance.

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As the approximation progresses, we see that the surface of the left plot aligns more closely with the surface of the right plot, confirming that He's method is effective in refining the solution step by step. The comparison between these two plots highlights the gradual convergence of the approximation towards the exact solution, demonstrating that He's method achieves high accuracy as the number of iterations increases.

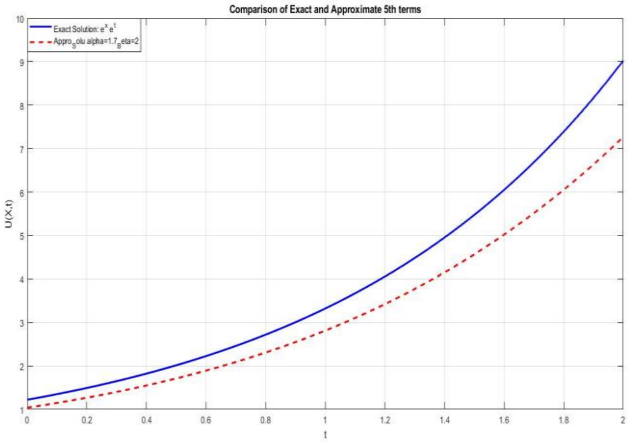

Fig. 2. Surface plots comparing the approximation and exact solution for $U(x,t)$ over the domains $0 \leq x \leq 10$ and $0 \leq t \leq 20$

Fig. 3 illustrates the comparison between the exact solution and the approximation for the fractional differential equation at $\alpha = 1.7$, $\beta = 2$, and after taking **5 terms** in the approximation process. Initially, the approximation curve deviates from the exact curve, particularly because α\alphaα is not an integer value. However, as the number of terms in the approximation increases, the gap between the approximation and exact solution begins to close.

The key observation here is that as α approaches 2, the approximation curve aligns perfectly with the exact solution curve. This behavior emphasizes the significance of integer values of α in the approximation process. When α becomes an integer (in this case, $\alpha = 2$), the approximation reaches the exact solution without further iterations, demonstrating the power of He's method in solving fractional partial differential equations.

At $\alpha = 1.7$, the fractional derivative introduces some deviations from the exact solution, which decreases as the approximation progresses. However, the convergence to the exact solution is much clearer as α\alphaα increases and approaches its integer value, where the behavior of the fractional derivative becomes more similar to its classical (integer) counterpart.

The fixed value of $\beta = 2$ ensures that the spatial derivative is also well-defined, which is why the approximation converges more easily when α\alphaα reaches an integer. This result underlines the effectiveness of He's method for approximating solutions to fractional partial differential equations, especially when the order of the fractional derivative becomes an integer.

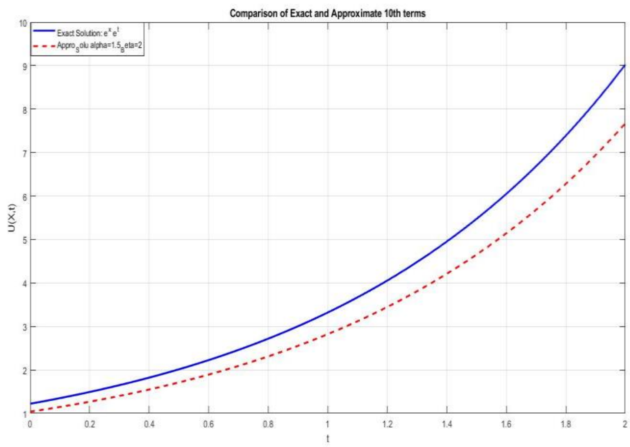

Fig 3. More terms give best approximation solution than Fig. 2, where fixed ($\alpha = 1.5$, $\beta = 2$)

Fig. 4 displays the comparison between the exact solution and the approximation for the fractional differential equation at $\alpha = 2$, $\beta = 2$, and after taking 7 terms in the approximation process. In this case, the approximation curve closely follows the exact solution curve, even after the 7th term.

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The critical observation here is that for integer values of α and β, specifically when $\alpha = 2$ and $\beta = 2$, the approximation curve quickly converges to the exact solution. This is a key feature of He's method: when the orders of the fractional derivatives become integers, the approximation method reaches the exact solution in just a few terms. In Fig. 4, after just 7 terms, the approximation curve is nearly indistinguishable from the exact solution.

For fractional partial differential equations, the number of terms required to achieve a high level of accuracy depends on the values of α and β. When α and β are both integers, the convergence of the approximation is fast and the approximation method can provide a solution that is practically identical to the exact one. This is demonstrated in Fig. 4, where the approximation curve is essentially a perfect match to the exact curve after 7 terms.

The fixed value of $\beta = 2$, which represents the spatial derivative, ensures that the approximation process is not influenced by spatial fractional order and focuses solely on the temporal behavior described by α. Given that $\alpha = 2$ corresponds to a classical (integer) derivative in time, the approximation method in this case performs similarly to solving a traditional differential equation without fractional derivatives, resulting in an exact match to the exact solution. One can see that approximation solution (curve) go to closed with exact solution when we taking more terms and at (α , β) go to 2.

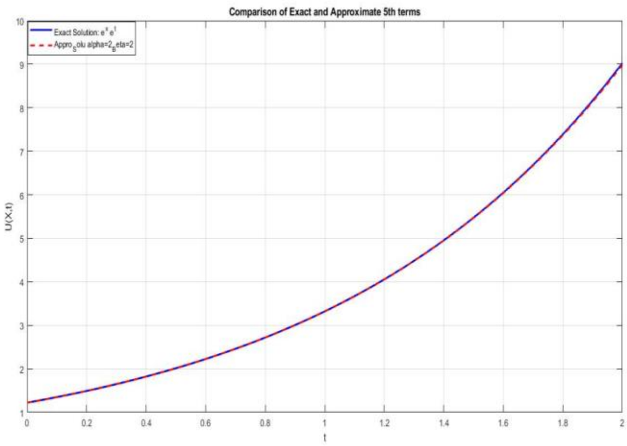

Fig. 4. The curves of exact and approximations solutions at $\alpha = 2, \beta = 2$ and taking 7th terms

EXAMPLE 2

Solve the fractional partial differential equation

${}_0^C D_t^{\alpha} \emptyset(x,t) = {}_0^C D_t^{\beta} \emptyset(x,t) + G(x,t)$, where $a \in R$, $0 < x < 1$, $t > 0$, with the following conditions $0 < \alpha \leq 1$ and $1 < \beta \leq 2$, $\emptyset(x,t) = (x^2 + t^{2a})$,

with $a = 1$, $G(x,t) = (\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{2x^{2-\beta}}{\Gamma(3-\beta)})$; respected to initial condition: $\emptyset(x,0) = x^2$.

Solution

He's method has been applying to solve this equation. The initial guess is selected as

$$\emptyset_0 (x,t) = I.C = x^2 \tag{57}$$

This choice aligns with the initial condition $\emptyset(x,0) = x^2$. The correction functions are iteratively calculated using He's method, which involves refining the solution at each step with corrections based on the fractional derivatives.

The general correction equation is given by

$$\emptyset_{n+1}(x,t) = \emptyset_n(x,t) - \int_0^t \left\{ D_s^{\alpha} \emptyset_n(x,s) - D_x^{\beta} \emptyset_n(x,s) - G(x,s) \right\} ds \tag{58}$$

where D_s^{α} and D_x^{β} represent the fractional derivatives in time and space, respectively.

Iterative Solutions:

1. **First Approximation ($n = 1$)**

$$\emptyset_1(x,t) = x^2 + \frac{2}{\Gamma(4-\alpha)} t^{(3-\alpha)} \tag{59}$$

2. **Second Approximation ($n = 2$)**

$$\emptyset_2(x,t) = x^2 + \frac{4}{\Gamma(4-\alpha)} t^{(3-\alpha)} - \frac{2}{\Gamma(5-2\alpha)} t^{(4-2\alpha)} \tag{60}$$

3. **Third Approximation ($n = 3$)**

$$\emptyset_3(x,t) = x^2 + \frac{6}{\Gamma(4-\alpha)} t^{(3-\alpha)} - \frac{6}{\Gamma(5-2\alpha)} t^{(4-2\alpha)} + \frac{2}{\Gamma(6-3\alpha)} t^{(5-3\alpha)} \tag{61}$$

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4. Fourth Approximation ($n = 4$)

$$\emptyset_4 = x^2 + \frac{8}{\Gamma(4-\alpha)}t^{(3-\alpha)} - \frac{8}{\Gamma(5-2\alpha)}t^{(4-2\alpha)} + \frac{4}{\Gamma(6-3\alpha)}t^{(5-3\alpha)} - \frac{2}{\Gamma(7-4\alpha)}t^{(6-4\alpha)} \quad (62)$$

5. Fifth Approximation ($n = 5$)

$$\emptyset_5 = x^2 + \frac{10}{\Gamma(4-\alpha)}t^{(3-\alpha)} - \frac{10}{\Gamma(5-2\alpha)}t^{(4-2\alpha)} + \frac{6}{\Gamma(6-3\alpha)}t^{(5-3\alpha)} - \frac{6}{\Gamma(7-4\alpha)}t^{(6-4\alpha)} + \frac{2}{\Gamma(8-5\alpha)}t^{(7-5\alpha)} \quad (63)$$

Exact Solution

If $\alpha = 1$ and $\beta = 2$, the approximation solutions become exact. In this case, the solution simplifies to:
$\emptyset_i(x,t) = x^2 + t^2$ for $i = 1,2,3,4,5$
Thus, the final solution is $\emptyset_n(x,t) = x^2 + t^2$, which matches the exact solution.

Results

Table 2 presents the results for different values of α and β, including the maximum errors between the exact solution and the numerical solutions for various numbers of terms in the approximation.

Table 2. Numerical solutions obtained using He's method for the fractional partial differential equation compared with the exact solution

Terms	Figure	X	t	α and β	Exact and Numerical Curves	Max-Error
3	A	0: 0.1: 1	$t = 0: 0.2: 2$	$\alpha = 0.3, \beta = 2$	Curves of exact and numerical solutions	0.3275
3	B	0: 0.1: 1	$t = 0: 0.2: 2$	$\alpha = 0.7, \beta = 2$	Curves of exact and numerical solutions	0.0415
3	C	0: 0.1: 1	$t = 0: 0.2: 2$	$\alpha = 1, \beta = 2$	Numerical = Exact solution	8.8818e-16
5	D	0: 0.1: 1	$t = 0: 0.2: 2$	$\alpha = 0.9, \beta = 2$	Curves of exact and numerical solutions	0.8032
5	E	0: 0.1: 1	$t = 0:.2: 2$	$\alpha = 1, \beta = 2$	Numerical = Exact solution	1.7764e-15

From Table 2, this table has the following columns, number of terms of approximation solution, fig, intervals of x and t, the different values of α and β and the maximum errors between both curves (approximation curves and the exact curve) respectively, in this table one can see that, we get small errors with more terms or in taking values of α. Near 1 with fixed β at 2, finally if $\alpha = 1$ and $\beta = 2$ we get exact solution.

Fig. 5 consists of five graphs that compare the exact solution with the numerical approximation for different values of α and β, as well as the number of terms used in the approximation. **Fig. 5A** presents the comparison for $\alpha = 0.3$ and $\beta = 2$, using 3 terms. In this case, the numerical solution deviates significantly from the exact solution, resulting in a large error. **Fig. 5B** shows the results for $\alpha = 0.7$ and $\beta = 2$ with 3 terms. As α approaches 1, the numerical solution starts to converge toward the exact solution, reducing the error. **Fig. 5C** illustrates the case for $\alpha = 1$ and $\beta = 2$, with 3 terms. Here, the numerical solution perfectly matches the exact solution, demonstrating that when both α and β are integer values, He's method provides the exact solution. **Fig. 5D** presents results for $\alpha = 0.9$ and $\beta = 2$ with 5 terms. Despite increasing the number of terms, the numerical solution still does not perfectly match the exact solution, indicating that non-integer values of α prevent exact convergence. Finally, **Fig. 5E** shows the results for $\alpha = 1$ and $\beta = 2$ with 5 terms, where the numerical solution aligns with the exact solution, confirming that with sufficient terms and integer values for α and β, He's method produces the exact solution.

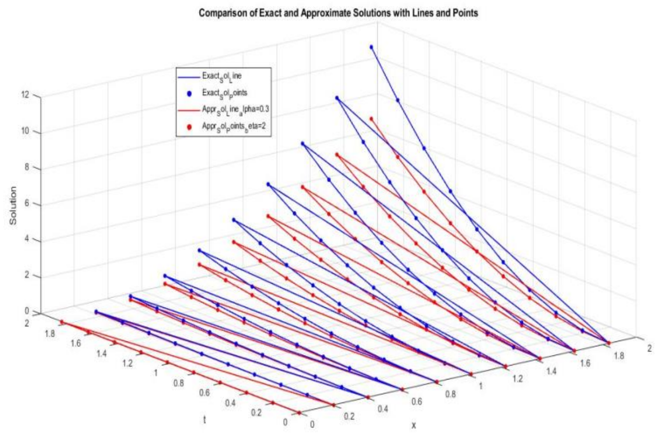

Fig. 5A. The curves of exact and approximations solutions at $\alpha = 0.3, \beta = 2$ and taking 5th terms

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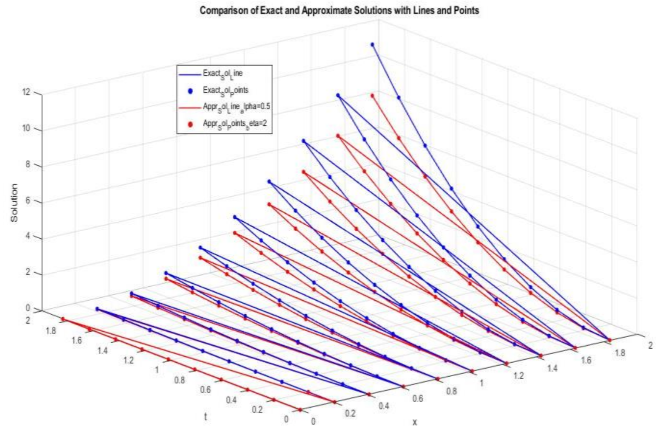

Fig. 5B. The curves of exact and approximations solutions at $\alpha = 0.5, \beta = 2$ and taking 5th terms

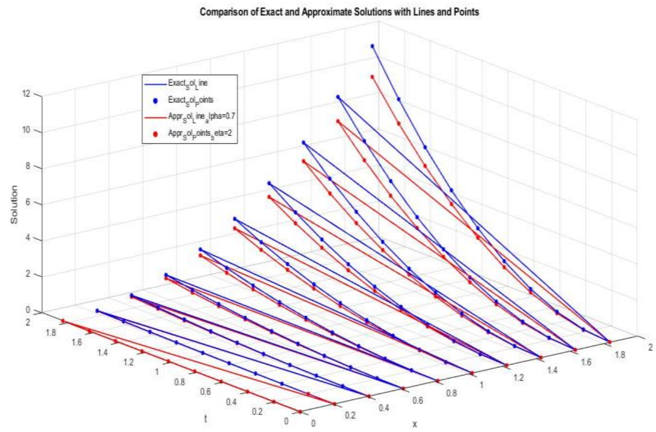

Fig. 5C. The curves of exact and approximations solutions at $\alpha = 0.7, \beta = 2$ and taking 5th terms

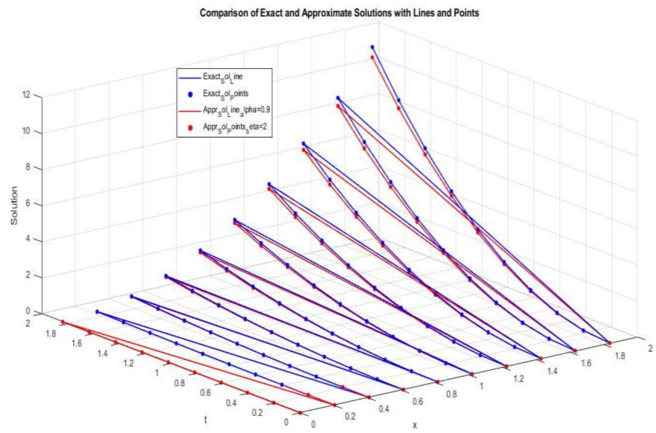

Fig. 5D. The curves of exact and approximations solutions at $\alpha = 0.9, \beta = 2$ and taking 5th terms

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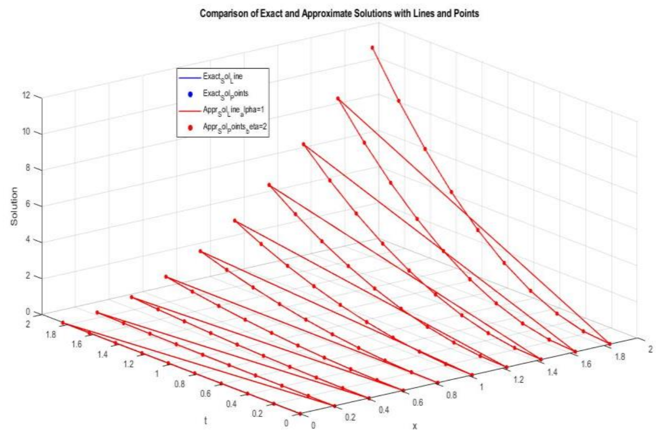

Fig. 5E. The curves of exact and approximations solutions at $\alpha = 1, \beta = 2$ and taking 5th terms

The analysis shows that **He's method** converges to the exact solution when $\alpha = 1$ and $\beta = 2$, and the accuracy improves with more terms in the approximation. The error decreases as α approaches 1, and the method becomes exact for integer values of α and β.

EXAMPLE 3

Consider like-heat equation

$\frac{\partial^\alpha}{\partial t^\alpha}\emptyset(x,t) = \frac{\partial^\beta}{\partial x^\beta}\emptyset(x,t) + G(x,t)$, where $a \in R$, $0 < \alpha \leq 1$ and $1 < \beta \leq 2$. The function $G(x,t)$ is given by $G(x,t) = a^\alpha x e^{at}$, and the initial condition is $\emptyset(x,0) = x,\ t > 0$.

Solution

Using **He's method**, we start by choosing the initial $\emptyset_0(x,t) = x$, which matches the initial condition $\emptyset(x,0) = x$. Since $0 < \alpha \leq 1$, the Lagrange multiplier λ is set to -1. The general correction equation is:

$$\emptyset_{n+1}(x,t) = \emptyset_n(x,t) - \int_0^t \lambda(x,s)\left\{D_s^\alpha\, \emptyset_n(x,s) - D_x^\beta\, \emptyset_n(x,s) - G(x,s)\right\}ds \tag{64}$$

Using this recursive method, we find the following approximations for $\emptyset(x,t)$ at different steps

1. **First approximation ($n = 1$)**

$$\emptyset_1(x,t) = xe^{at} + a^{(\alpha-1)} \tag{65}$$

2. **Second approximation ($n = 2$)**

$$\emptyset_2(x,t) = xe^{at}\left(1 - 2a^{(\alpha-1)} + a^2(\alpha-1)\right) + xe^{at}(2a^{(\alpha-1)} - a^2(\alpha-1)) \tag{66}$$

3. **Third approximation ($n = 3$)**

$$\emptyset_3(x,t) = xe^{at}\left(1 - 3a^{(\alpha-1)} + a^2(\alpha-1) + a^3(\alpha-1)\right) + xe^{at}(3a^{(\alpha-1)} - a^2(\alpha-1) - a^3(\alpha-1)) \tag{67}$$

4. **General form for the $n - th$ approximation**

$$\emptyset_n(x,t) = xe^{at}\left(1 - na^{(\alpha-1)} + a^2(\alpha-1) + \cdots + a^n(\alpha-1)\right) + xe^{at}(na^{(\alpha-1)} - a^2(\alpha-1) - \cdots - a^n(\alpha-1)) \tag{68}$$

Exact solution

When $a = 1$, $\alpha = 1$, and $\beta = 2$, the approximation converges to the exact solution:

$$\emptyset(x,t) = \emptyset_n(x,t) = xe^{at} \tag{69}$$

This exact solution satisfies the given fractional partial differential equation. Therefore, as the number of iterations n approaches infinity, the approximation becomes the exact solution.

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Fig. 6A, **B** and **C** show the approximation solution at $(x = 0.2, t = linspace(0,2,25)$ with 8 terms and $\alpha = 0.5$, 0.8 and 1 for $\beta = 2$, one can see when $\alpha = 1$ and $\beta = 2$, $\alpha = 1$ the approximation solution closed with exact solution.

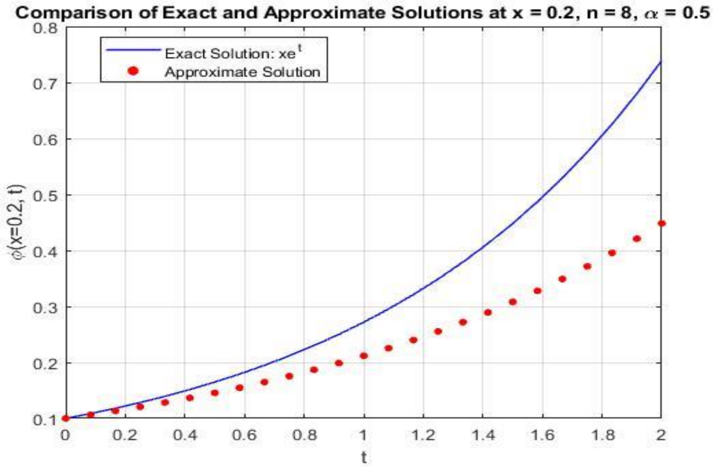

Fig. 6A. Approximation solution for $x = 0.2$, $t = linspace(0, 2, 25)$, with 8 terms and $\alpha = 0.5$, $\beta = 2$

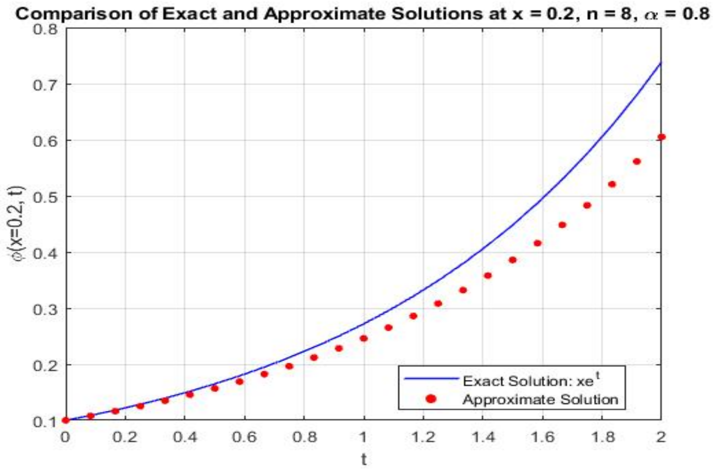

Fig. 6B. Approximation solution for $x = 0.2$, $t = linspace(0, 2, 25)$, with 8 terms and $\alpha = 0.8$, $\beta = 2$

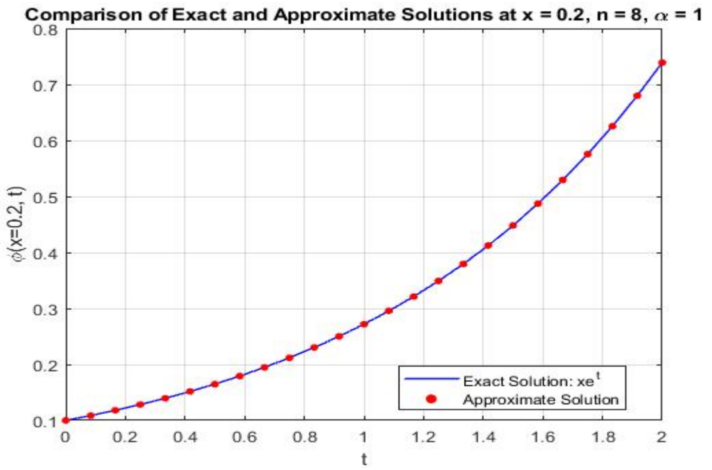

Fig. 6C. Approximation solution for $x = 0.2$, $t = linspace(0, 2, 25)$, with 8 terms and $\alpha = 1$, $\beta = 2$

- **He's method** converges to the exact solution $\emptyset(x,t) = xe^{at}$ as the number of terms in the approximation increases.
- For integer values of α and β (i.e., when $\alpha = 1$ and $\beta = 2$), the approximation coincides with the exact solution after a few iterations.
- For non-integer values of α, the approximation improves with each additional term but does not exactly match the exact solution until the iteration count is very high or α becomes close to 1.

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CONCLUSION

In this paper, the algorithmic steps of He's method and applied it to three different fractional partial differential equations has been discussed. Several key observations emerged from the study. Firstly, He's method proves to be versatile and easy to apply to a wide range of fractional differential equations. It provides accurate approximation solutions, and importantly, it allows for the determination of exact solutions when needed. The general steps of the method were clearly outlined, showcasing its systematic approach. From the results obtained in the three examples, as well as from the tables and figures presented, we observed that the maximum errors decreased significantly when more terms of the approximation were included. Additionally, when the values of the fractional orders α and β were taken closer to integer values, the error was minimized. Our discovery was that a fractional partial differential equation of the first kind with the fractional derivatives are driven to an equation of whole numbers i.e. when $\alpha = 1$ and $\beta = 2$. The finding actually confirms the power of the method.

This method described in this paper is efficient because it is based on a correcting sequence where correction equations are used to gradually improve the numerical solution. This recursive relation generates the sequence by which the approximate solutions are computed. The approach also allows the one to choose the first approximations for the solutions which are the first steps in solving the problems. As an addition, the method is computationally effective and easily implementable by using computer programming tools like python which was used in this study. This is one of the main reasons why He's method is an attractive feature in the practical applications like fractional-order differential equations.

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