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Solving fractional differential equations by means of differential matrices based on the modified Legendre polynomials

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ABSTRACT

Fractional calculus is another variant of classical calculus that enriches the concept of differentiation and integration by allowing orders that are non-integer numbers. This generalization makes it possible to formulate fractional differential equations (FDEs) that can describe a wider range of phenomena which could hardly be modeled by integer-order differential equations. Due to the complexity and often the impossibility of obtaining analytical solutions for FDEs, numerical methods have become crucial means to solve these equations.

In this paper, we introduce a simple and efficient numerical method to solve FDEs with the aid of a matrix system, making it particularly suitable for the challenges of fractional derivatives handling. The proposed method incorporates modified orthogonal polynomials, which are specifically designed to match the problem's concerns. Those polynomials are tuned in order to provide a high level of accuracy and fast convergence during the process of approximation of the solution of the FDE. The convergence behavior of the proposed method is thoroughly explored in order to evaluate the efficiency for different types of fractional differential equations.

To show the practical application of this method and other, we present some examples by themselves, explaining how the solutions to the FDEs are obtained for various values of the fractional derivatives. The results are presented in the shape of tables, so the readers can make a comparison of the accuracy and performance of the method under different conditions. The paper sets out to prove this by demonstrating the pros of this new way of carrying out the task and thus allow researchers and practitioners to quickly and effectively obtain numerical solutions for such equations.

Keywords: Time-fractional differential equations, numerical methods, modified orthogonal polynomials, matrix system, convergence analysis

INTRODUCTION

The theme of this is the job of Fractional Differentiation Equations (FDE) that are an expansion of classical ordinary differential equations in which the derivatives are generated with the order of a non-integer, often fractional calculus being used [1]. The questions have been serving a critical role in many branches of science and engineering because they can forecast complex phenomena where integer-order differential equations won't do [2]. The subject of FDEs touches various spheres, e.g, physics, biology, control theory, and finance, where the systems possess memory effects, long-range interactions, and anomalous diffusion [3].

Finding an analytical solution for the FDEs mostly is a problem, as closed-form solutions are the case for only rare cases [4]. For this reason, the systems are frequently approached through numerical methods to approximate the solutions of FDEs [5]. Of all the existing numerical methods, the use of polynomial approximations has lately become an effective tool to approach the FDE problem [6]. The Modified Lagrange polynomials are on the list of the various fractional differential equations' numerical solutions that have been quite effectively solved by their numerical methods [7].

The Lagrange polynomials are a well-established family of interpolation polynomials that can be made use of in the approximation of the functions for some certain number of the given data points [8]. The modified Lagrange polynomial method, on the other hand, brings in some modifications that improve the regular Lagrange polynomials' accuracy and power of computation [9]. The algorithm involves manipulating the common Lagrange basis polynomials to suit the special case of partial-order derivatives appearing in the given FDEs [10].

Essentially, the differential matrix of the modified Lagrange polynomials is built by modeling a matrix representation of the derivatives of the modified Lagrange polynomials [11]. These polynomials can be employed to achieve the solution of an FDE by

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transforming it into a system of algebraic equations through discretization of the equation [12]. Then, this matrix derived can be solved by standard methods for approximate solution of the FDE [13]. The differential matrix does in actual fact change a fractional differential equation into a set of linear equations of which can be operated computationally by a computer [14].

Among the many contributions given by the modified Lagrange polynomials in this context, one of the most important is that they can approximate the fractional derivatives highly accurately when using rather coarse grids or dealing with a small number of data points. [15]. Furthermore, they provide a means of addressing the problem in conditions given by the boundary sets, initial conditions, and the input value of h which would not be capable of being solved with the earlier approach. Additionally, the methodology allows for the presence of the nonlinearity in fractional differential equations, which is found in most practical systems [16].

Differential matrix makes use of the modified Lagrange pertaining to all the advantages aforementioned and hence lends itself to only fewer computations which in essence happens when all of them build up the superstructure. For example, the assembly of the matrix is pretty easy and the resulting system of equations is sparse simplifying the work of whoever the numerical solver is. Also, the method grants a possibility for both time and space fractional derivatives to be put into a process where both time and space are fractional [17].

The use of the differential matrix of modified Lagrange polynomials is an efficient and accurate numerical tool for solving fractional differential equations [18] in this context. This technique has demonstrated its reliability in a wide range of fractional models [19]; thus, it is a great tool for researchers and practitioners in those areas where fractional differential equations are employed. The flexibility and computational advantages that come with this approach are the main reasons for its adoption in different fields of study [20].

Fractional differential equations (FDEs) have proven to be a powerful tool for modeling systems that have memory effects, long-range interactions, and anomalous diffusion. However, analytically solving FDEs can be quite a challenge because of the complex nature of fractional derivatives. Usually, numerical methods are the ones that are used to approximate solutions; however, the methods that are already in place are either not as they should be or simply are not accurate enough. The current way is to create a more efficient and accurate numerical approach to solve FDEs that can handle fractional derivatives of arbitrary order and be quite a computationally reasonable manner.

Even though there have been many numerical approaches brought up to tackle FDEs, it is still going to be difficult to find the ones that effectively handle fractional derivatives and give high accuracy, for most with a maximal computational resource use. Many of these approaches like finite difference or spectral methods may fail to converge or would need high computational resources to achieve high accuracy. This research looks forward to presenting a matrix system solution approach that is based on the use of modified orthogonal polynomials with better accuracy and efficiency. This work outlines a new method, involving a matrix system built based on modified orthogonal polynomials, that will improve the accuracy and efficiency of the solution of fractional differential equations. In this study, the main goals are trying this method on fractional differential equations of different types, looking for its convergence behavior, and comparing through the analysis of it with the methods.

This study's objective is the development of a numerical method that efficiently solves fractional differential equations (FDEs) while maintaining high accuracy. The through this the to implement a matrix system that uses modified orthogonal polynomials for approximating the solution of FDEs was the purpose of the research. This method is believed to promote computational efficiency by treating the complexity of fractional derivatives. At the same time the method aims at analyzing the convergence behavior to makes sure the method is reliable and efficient under the possibility of several different scenarios. The steps to make the practical impact of the method will be examples of solving fractional parts and giving of different types of boundary conditions using different methods. The suggested problems are to represent the method as a standard tool to solve various types of FDEs and to check if the method overcomes the existing numerical methods.

This research bears resemblance to prior works that inspired by polynomial approximation methods applied for solving first and second order differential equations. Similarly to some of the applications using the other basis functions, this method also employs the polynomial interpolation in redirecting the solutions to fractional differential equations. Moreover, the developments concerning matrix-based systems for the treatment of fractional derivatives have been of great help, especially in the area of several numerical methods.

Being the foundation of this article, we can locate the innovative part in two aspects that can be summarized as the combination of a matrix system and of such special orthogonal polynomials that have been derived for the treatment of the frequency problem. As usual, polynomials have been utilized for solving FDEs in the past. However, the present study has brought modifications to the conventional polynomial bases, making them more suitable and hence, achieving higher convergence and computational performance. Furthermore, the approach is designed to handle the derivatives of any order, which provides a wider range of applications when a fractional arbitrary model is adopted.

Owning to the additional developments, the paper underscores the proposal of a numerical framework that is simple to be implemented and the framework is intended for application to various fractional calculus problems. This method can be combined with other numerical solvers for a wider area of application in physics, biology, and engineering. With this approach, one can find new ways of polynomial-based methods for FDEs including a dynamic approach that modifies the polynomial degree according to the problem's complexity.

Even though the proposed method has been observed to work quite well, it does have some limitations. One of the main factors that will determine the correct working of the method is the choice of the polynomial degree, which might need tuning of different problems. Thus, the method's efficiency may be influenced by the boundary conditions and the type of fractional derivatives utilized. The future research should be directed towards the development of the technique that can be utilized to enhance the method's robustness by adopting adaptive techniques for the selection of the polynomial degree and the method to be applied to more complex fractional models. Also, the method is applicable to nonlinear fractional differential equations that remain an open area for future exploration.

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The biggest problems we face in solving FDEs by numerical methods come from the computation of fractional derivatives and ensuring the convergence of the numerical method. The classical models are usually restrictive in terms of the slow asthmatic convergence or examples of a serious increase of computational expenses in the case of activities including fractional orders. The modeling of this research breaks through as it provides a numerical scheme based on the basic idea of polynomials, enabling the exact solutions with exact efficiency. Nevertheless, more progress should be made in order for the method to be able to deal with nonlinear FDEs, multidimensional fractional models, and irregular domains.

SPECIAL FUNCTIONS: GAMMA FUNCTION

One of the basic functions in fractional arithmetic is the gamma function, which has the following definition and some properties [21]

1. $\Gamma(n) = \int_0^M x^{n-1} e^{-x} \, dx \quad ; \qquad n > 0, x \in R,$ (1)
2. $\Gamma(x+1) = x\Gamma(x) \qquad \forall x \neq 0,$ (2)
3. $\Gamma(n+1) = n!$ for integer $n \geq 0$, (3)
4. $\Gamma(n)\Gamma(n+\frac{1}{2}) = 2^{1-2n}\sqrt{\pi}\,\Gamma(2n)$ (4)
5. $\Gamma\left(n+\frac{1}{2}\right) = \frac{1\times3\times5\times\ldots\times(2n-1)}{2^n}\sqrt{\pi},\ n = 1,2,3,\ldots,$ (5)
6. $\Gamma(-n+\frac{1}{2}) = \frac{(-1)^n 2^n \sqrt{\pi}}{1\times3\times5\times\ldots\times(2n-1)2^n}$ (6)
7. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$ (7)

Legendre polynomial: it given by the form:

$$P_i(x) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!x^k}{(i-k)!(k!)^2} \tag{8}$$

It is orthogonal respect with Wight function $u^{0,0}(x) = 1$.

To work with fractional derivatives and integrals, the properties become crucial in expressing fractional orders of derivatives in terms of the Gamma function.

FRACTIONAL DERIVATIVES AND FRACTIONAL INTEGRALS

Fractional calculus, the basis of which is traditional calculus, can be extended to differentiation and integration orders that are not natural numbers. There are many non-local interactions, memory effects, or anomalous diffusion systems where this approach makes sense. On the other hand, traditional calculus of fractions, the extension of real numbers from traditional calculus, is incapable of describing the above-mentioned effects accurately. It has been shown that fractional differential equations (FDEs) are to be found in high-speed electronics, medicine, physics, and finance. However, it is not easy to solve them analytically due to the non-integer order of derivatives. The use of numerical methods, such as the Legendre polynomial, results in an effective way to address such problems [22].

This paper begins with the presentation the defining Becker's codings of fractional derivatives and integrals, their properties, and the constructions of Legendre polynomials in the relativistic calculus [3]. Later, it describes the role of Legendre polynomial in approximating solutions to the FDEs [4].

CONVERGENCE ANALYSIS

we know that, the Legendre polynomials $(n = 6)$, $P_i(x)$, it be complete set from $L_2[0,1]$ orthogonal, then since $H^m[0,1]$ it is soperlive space for all $u(t)$ on the interval $[0,1]$ within all derivatives of degree m in $L_2[0,1]$.

The definition of $\|.\|_{H^m([0,1])}$ given as:

$$\|u\|_{H^m(\Omega)} = \left(\sum_{k=0}^{m}\|u^{(k)}(t)\|^2_{L^2(\Omega)}\right)^{1/2} \tag{9}$$

also the half of this norm given as:

$$|u|^2_{H^{m:N}(\Omega)} = \left(\sum_{iinin(m,N)}^{N}\|u^{(i)}(t)\|^2_{L^2(\Omega)}\right) \tag{10}$$

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one can see that for $N \geq m-1$, we have $|u|_{H^{m:N}(\Omega)} = \|u^m\|_{L^2(\Omega)} = |u|_{L^m(\Omega)}$ (11)

Let $u_N = \sum_{k=0}^{N} \widetilde{u_k} P_k$ is the approximation Legendre for $u \in H^m(\Omega)$, so the error is

$\|u - u_N\|_{H^{m:N}(\Omega)} = \|u^m\|_{L^2(\Omega)} \leq CN^{-m}|u|_{L^m(\Omega)}$, where C is positive constant. (12)

Fractional order differential equations (FDE's)

FDEs are the generated of the integer order, it have the general form:

$$P_n y(x) = \sum_{i=1}^{k} a_i D^{\alpha_i} y(x) + a_{k+1} y(x) + g(x) \tag{13}$$

Respect to the initial conditions $y^{(i)}(0) = d_i$, $i = 0, 1, 2, \dots, n-1$ (14)

where the coefficients $a_j (j = 0,1 \dots, k+1)$, are real constant.

Also, $0 < \beta_1 < \beta_1 < \cdots < \beta_k < \alpha$, $n - 1 < \alpha \leq n$, and D^α denoted the R_L FDEs with order α.

Legendre Matrix for Fractional order derivatives

$D^\alpha P_i(x) = 0$, $i = 0,1, \dots, \lceil \alpha \rceil - 1$, $\alpha > 0$, $P_i(x)$ is Legendre polynomial in (8),

For approximate unknown function $Y(x) = \sum_{j=0}^{\infty} c_j P_j(x)$,

where, the coefficients c_j given by $c_j = (2j+1) \int_0^1 Y(x) P_j(x) dx$, j=1,2,… ,

where the first (m+1) terms of Legendre polynomial (6) given by:

$Y(x) = \sum_{j=0}^{m} c_j P_j(x) = C^T \Phi(x)$,

where C the coefficients basis of Legendre polynomial and $\Phi(x)$ is the basis of Legendre, both given as: $C^T = [c_0, c_1, \dots, c_m]$,

and $\Phi(x) = [P_0(x), P_1(x), \dots, P_m(x)]^T$

The derivative of the vector $\Phi(x)$ given as $\frac{d\Phi(x)}{dx} = D^{(1)} \Phi(x)$, (15)

where $D^{(1)}$ is a $(m+1) \times (m+1)$, matrix ofderivatives operations given as:

$$D^{(1)} = d_{ij} = \begin{cases} 2(2j+1), \text{for } j = i - k \\ 0 \qquad \text{otherwise} \end{cases} \begin{cases} k = 1,2, \dots\dots m \quad \text{for m odd} \\ k = 1,3, \dots, m-1 \text{ for m even'} \end{cases}$$

For example: for m, we have $D^{(1)} = 2 \begin{bmatrix} 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0\cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0\cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0\dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0^{\cdots} & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7^{\cdots} & 0 & 2m-1 & 0 \end{bmatrix}$,

This matrix is a matrix of Legendre relations for fractional calculating, by using (15), we can get, $\frac{d^n \Phi(x)}{dx^n} = \left(D^{(1)}\right)^n \Phi(x)$, where $n \in \mathbb{N}$, So that $D^{(n)} = \left(D^{(1)}\right)^n$, $n = 1,2, \dots$

Now to generate the matrix of Legendre differential operations (15), for fractional derivatives:

Let $\Phi(x)$ be the basis for Legendre polynomials (14), and let $\alpha > 0$, so that:

$$D^\alpha \Phi(x) = D^{(\alpha)} \Phi(x), \tag{16}$$

where $D^{(\alpha)}$ is the matrix of operators for fractional differential, with the size $(m+1)(m+1)$ by Caputo fractional derivatives of order α ,

This has the form:

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$$D^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} \end{pmatrix} \tag{17}$$

where $\theta_{i,j,k}$ given by the form:

$$\theta_{i,j,k} = (2j+1)\sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l-\alpha+1)} \tag{18}$$

To prove this form: by using analytic Lagrange polynomial (8) by Cabuto fractional derivative, also by linearity propriety of fractional operator we can get:

$$D^{\alpha}P_i(x) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{\alpha}\left(x^k\right),\ \ i = \lceil\alpha\rceil, \dots, m \tag{19}$$

For $f(x) = x^k$, we have $D^{\alpha}f(x) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}x^{k-\alpha} = \frac{k!}{\Gamma(k+1-\alpha)}x^{k-\alpha}$ (20)

Eq. (19) will be, $D^{\alpha}P_i(x) = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)}x^{k-\alpha}, i = \lceil\alpha\rceil, \dots, m$ (21)

If we approximate $x^{k-\alpha}$ by $(m+1)$ conditions of Legendre (6) to get:

$x^{k-\alpha} \cong \sum_{j=0}^{m} b_{k,j}P_j(x)$, where $b_{k,j}$ given by: (22)

$$b_{k,j} = (2j+1)\int_0^1 x^{k-\alpha}P_j(x)dx = (2j+1)\sum_{i=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2}\int_0^1 x^{k+l-\alpha}dx = (2j+1)\sum_{i=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(k+l-\alpha+1)}, \tag{23}$$

using Eq. (19 –21); we can get:

$$D^{\alpha}P_i(x) \cong D^{\alpha}P_i(x) = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!\,(k!)\Gamma(k+1-\alpha)} \sum_{j=0}^{m} b_{k,j}P_j(x)$$

$$= \sum_{k=\lceil\alpha\rceil}^{i}\sum_{j=0}^{m} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)} b_{k,j}P_j$$

$$= \sum_{k=\lceil\alpha\rceil}^{i}\sum_{j=0}^{m} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)} (2j+1)\sum_{i=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(k+l-\alpha+1)},$$

$$= \sum_{k=\lceil\alpha\rceil}^{i}\sum_{j=0}^{m}(2j+1)\sum_{i=0}^{j} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)} \frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(k+l-\alpha+1)},$$

$$= \sum_{j=0}^{m}\sum_{k=\lceil\alpha\rceil}^{i}(2j+1)\sum_{i=0}^{j} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-\alpha)} \frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(k+l-\alpha+1)}$$

$$=\sum_{j=0}^{m}\left(\sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,j,k}\right)P_j(x)\ \ ,\ i = \lceil\alpha\rceil, \dots, m\ , \tag{24}$$

where $\theta_{i,j,k}$ given by (18), we can rewrite (24) as a vector:

$$D^{\alpha}P_i(x) \cong \left[\sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,0,k}, \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,1,k}, \dots, \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,m,k}\right]\Phi(x), i = \lceil\alpha\rceil, \dots, m \tag{25}$$

Also, we can write $D^{\alpha}P_i(x) = [0,0,\dots,0]\ \Phi(x)$ where $i = 0,1,\dots\lceil\alpha\rceil - 1$, (26)

so that both (24-25) equations will make proof done.

Solving Linear FDE: the linear FDE's given by:

$$D^{\alpha}y(x) = a_1D^{\beta_1}y(x) + \cdots + a_kD^{\beta_k}y(x) + a_{k+1}y(x) + a_{k+2}g(x) \tag{27}$$

respect with initial conditions: $y^{(i)}(0) = d_{i}, i = 0,1,\dots,n$, (28)

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where $j = 1, \dots, k + 2, \alpha_i$,also, $\beta_1 < \beta_2 < \cdots < \beta_k < \alpha$, where D^α denote the Caputo fractional derivative of order α.

Now, to find solution $y(x)$, we approximate $y(x)$ and $g(x)$ by Legendre (8) as:

$y(x) \cong \sum_{i=0}^{m} c_i P_i(x) = C^T \Phi(x)$ (29)

$g(x) \cong \sum_{i=0}^{m} g_i P_i(x) = G^T \Phi(x)$ (30)

where vector $G^T = [g_0, g_1, \dots, g_m]^T$, known, $C^T = [c_0, c_1, \dots, c_m]^T$ unknown, (31)

using Eqns. (16-27); we get $D^\alpha y(x) \cong C^T D^\alpha \Phi(x) \cong C^T D^{(\alpha)} \Phi(x)$ (32)

also $D^{\beta_j} y(x) \cong C^T D^{\beta_j} \Phi(x) \cong C^T D^{(\beta_j)} \Phi(x)$ where $j = 1, \dots, k$, (33)

by using (27-28) we can find Remained $R_m(x)$, we can write:

$R_m(x) \cong \left(C^T D^{(\alpha)} - C^T \sum_{j=1}^{k} a_j D^{(\beta_j)} - a_{k+1} C^T - a_{k+2} G^T \right) \Phi(x)$ (34)

By the same way in tau method [42], we make linear equation (m-n) by applying

$\langle R_m(x) - P_j(x) \rangle = \int_0^1 R_m(x) P_j(x) dx = 0$, where $j = 0,1, \dots, m - n - 1$, (35)

Also we have $(0) = C^T \Phi(0) = d_0$,

$y^{(1)}(0) = C^T D^{(1)} \Phi(0) = d_1$,

$y^{(n)}(0) = C^T D^{(n)} \Phi(0) = d_n$ (36)

SOLVE NONLINEAR FDE'S

This kinds has the general form;

$D^\alpha y(x) = F(x, y(x), D^{\beta_1} y(x), \dots, D^{\beta_k} y(x))$ (37)

with I.C $y^{(i)}(0) = d_i$, $i = 0,1, \dots, n$. (38)

where $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$, $n < \alpha \le n + 1$, also D^α denoted the Caputo Fractional derivative of order α (F may be not linear in general case).

Now we approximate $(y(x), D^\alpha y(x), D^{\beta_i} y(x))$ for $= 0, .., k$, using basis of Legendre (8), to get

$C^T D^{(\alpha)} \Phi(x) = \left(x, C^T \Phi(x), C^T D^{(\beta_i)} \Phi(x), \dots, C^T D^{(\beta_k)} \Phi(x) \right)$ (39)

Also by use Eqns. 15-16 in Eq. 25 we get:

$y(0) = C^T \Phi(0) = d_0$ (40)

$y^{(i)}(0) = C^T D^{(i)} \Phi(0) = d_i$, $i = 1,2, \dots, n$ (41)

Now by solving the system of these nonlinear equations and fixe the coefficients C, we can calculate the solution $y(x)$, as in Eq. 25.

RESULTS AND DISCUSSION

Three different examples to show the powerful of this method, where in the first one we get the exact solution, in second

$(0 < \alpha \le 1)$ and in third $(1 < \alpha \le 2)$ we will get the approximation solution which is closed with exact when fractional derivative equat integer derivative, as we show below,

EXAMPLE 1

Solve the non-homogenous FDE, $D^2 y(x) + D^{3/2} y(x) + y(x) = 1 + x$ (42)

Respecte to I.C $y(0) = 1, y'(0) = 1$, where exact solution is $(x) = 1 + x$ (43)

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Solution for $\mathrm{m} = 2$ the approximate solution will be given as:

$$y(x) = c_0P_0(x) + \ c_0P_0(x) + + c_0P_0(x) = C^T\Phi(x) \tag{44}$$

So that by using matrix as;

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}, \tag{45}$$

$$D^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{pmatrix}, \tag{46}$$

$$D^{(3/2)} = \left(\frac{16}{\sqrt{\pi}}\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3/5 & 1/7 \end{pmatrix}, \tag{47}$$

$$\text{and} \quad G = \begin{pmatrix} 3/2 \\ 1/2 \\ 0 \end{pmatrix}, \tag{48}$$

Now by using the form (first solve FDE), we get:

$$c_0 + \left(12 + \frac{16}{\sqrt{\pi}}\right)c_2 - \frac{3}{2} = 0, \tag{49}$$

Now by using $[y^{(n)}(0) = C^TD^{(n)}\Phi(0) = d_n]$, (50)

We get

$$c_0 - c_1 + c_2 - 1 = 0, \text{ and } 2c_1 - 6c_2 - 1 = 0, \tag{51}$$

Now by solving the two equations

$$c_0 = \frac{3}{2}, c_1 = \frac{1}{2}, c_2 = 0, \tag{52}$$

So that we can write the matrices of solution as:

$$y(x) = \left(\frac{1}{3,1}/2,0\right)\begin{pmatrix} 1 \\ 2x - 1 \\ 6x^2 - 6x + 1 \end{pmatrix} = 1 + x \tag{53}$$

This solution is exact solution.

EXAMPLE 2

Consider LFDE $D^{\alpha}y(x) + y(x) = 0$, where $0 < \alpha \leq 1$, respect to $\mathrm{I.C}\ y(0) = 1$, where the exact solution when $(\alpha = 1)$ is $y(x) = \exp(-x)$

By the same steps in example 1 we get the approximation solution by the formula

$$y(x) = \sum_{k=0}^{\infty} y_k = \sum_{k=0}^{\infty} \frac{(-x^{\alpha})^k}{\Gamma(\alpha k+1)}, \tag{54}$$

Table 1 provides the absolute error between the exact solution and the approximation for various values of xxx and for different numbers of polynomial terms in the expansion, denoted by N (where N=2, 4, 8). The table illustrates how the error decreases as N increases, indicating better accuracy of the approximation as more terms are included in the Legendre polynomial series.

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Table 1. The absolute error between exact and approximation solutions for different value of $(\alpha = 0.5)$

x	$N = 2$	$N = 5$	$N = 8$
0.1	4.8374e-03	1.6258e-04	8.1964e-08
0.2	1.8731e-02	1.2692e-03	2.5803e-06
0.3	4.0818e-02	4.1818e-03	1.9279e-05
0.4	7.0320e-02	9.6800e-03	7.9954e-05
0.5	1.0653e-01	1.8469e-02	2.4017e-04
0.6	1.4881e-01	3.1188e-02	5.8836e-04
0.7	1.9659e-01	4.8415e-02	1.2522e-03
0.8	2.4933e-01	7.0671e-02	2.4044e-03
0.9	3.0657e-01	9.8430e-02	4.2678e-03
1.0	3.6788e-01	1.3212e-01	7.1206e-03

From Table 1, the values presented in the table correspond to the absolute error for a fixed fractional order α = 0.5. The rows represent different values of x (ranging from 0.1 to 1.0), while the columns show the errors for different values of N, which are 2, 5, and 8.

Smaller Errors with Larger N

As N increases, the absolute error between the exact and the approximate solutions decreases significantly. This trend can be observed across all values of x. For instance, at $x = 0.1$, the error for $N = 2$ is 4.8374×10^{-3}, but when $N = 8$, the error reduces to 8.1964×10^{-8}, a significant reduction by several orders of magnitude. This behavior is consistent for all x values.

Effect of x

As the value of x increases, the absolute error generally increases for all N values. For example, at $x = 1.0$, the error for $N = 2$ is 3.6788×10^{-1}, which is much larger than the error at $x = 0.1$, where the error for $N = 2$ is 4.8374×10^{-3}. This trend suggests that the approximation method performs better for smaller values of x, which is common for numerical approximations involving polynomial expansions.

Convergence Behavior

The data clearly demonstrates that as N increases, the approximation converges to the exact solution. For $N = 2$, the error is relatively high across all x values, but by $N = 8$, the error is minimal, indicating that the method is converging towards the true solution as more terms are added to the polynomial expansion.

From the Table 1, we can conclude that the proposed method for solving fractional differential equations using Legendre polynomials is effective in reducing the approximation error as the number of terms in the expansion increases. The absolute error is noticeably smaller for larger N, with the approximation becoming more accurate as N grows. On the other hand, it swells at a more accelerated rate with large x, but the mistake is nevertheless there, yet its presence diminishes perceptibly with rising (N). We can therefore conclude that this algorithm is not only reliable but also efficient at getting good enough approximations for FDEs, and we can reach perfection by including higher-order polynomial expansions.

Fig. 1 shows how distant the exact and approximation solutions are, for various values of $N\ (2, 5, and\ 8)$ for $\alpha = 0.5$, and different values of x. The graph clearly demonstrates that as the number of terms (N) increases, the error decreases significantly, confirming the convergence of the approximation method. For smaller x values, the error is low even for smaller N, but as x increases, the error grows. However, for larger values of N, the error is minimized across all x.

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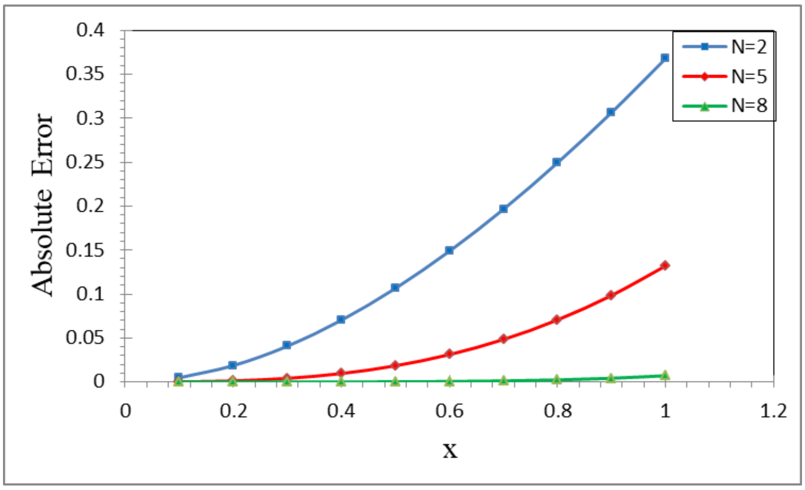

Fig. 1. Absolute error for different N values ($\alpha = 0.5$)

Table 2 presents the error between the exact solution $e^{-x}$and its approximation for various fractional orders of differentiation, $\alpha = 0.5, 0.7, 0.9, and\ 1$, at different values of x. The approximation is based on a fixed number of terms $N = 10$ in the polynomial expansion.

Decreasing Error as $\alpha \to 1$

As α approaches 1, the error between the exact and approximate solutions decreases significantly. For ($\alpha = 1$, the error is extremely small, often approaching machine precision (on the order of 10^{-16} or smaller), which is consistent with the fact that when $\alpha = 1$, the approximation closely matches the exact solution. For example, at $x = 0.1$, the error for $\alpha = 1$ is 1.1102×10^{-16}, effectively zero in the numerical sense.

Larger Errors for Smaller α

For smaller values of α, such as $\alpha = 0.5$, the error is much larger, especially for smaller values of x. At $x = 0.1$, the error for $\alpha = 0.5$ is 1.3693×10^{-9}, which is still very small but noticeably higher than the error for $\alpha = 1$. As (α increases, the error gradually decreases, indicating that the method performs better for larger fractional orders.

Effect of x

Increasing Error with Larger x

As the value of x increases, the absolute error tends to increase for all values of α. For example, at $x = 1.0$, the error for $\alpha = 0.5$ is 1.2128×10^{-3}, while for $\alpha = 1$, the error is 2.3114×10^{-8}. This pattern suggests that the approximation method becomes less accurate as x increases, particularly for smaller values of α.

For $\alpha = 0.5$

The errors are relatively large, particularly at smaller x values. For instance, at $x = 0.1$, the error is 1.3693×10^{-9}, which increases as x grows.

For $\alpha = 0.7$

The errors are smaller compared to $\alpha = 0.5$, but still noticeable. The error at $x = 0.1$ is 4.0646×10^{-2}, and it continues to decrease with increasing x.

For $\alpha = 0.9$

The errors are smaller than for $\alpha = 0.5$ and $\alpha = 0.7$. At $x = 0.1$, the error is 2.1955×10^{-2}, and it decreases with larger x, but remains higher than for $\alpha = 1$.

For $\alpha = 1$

The errors are extremely small, and as $\alpha \to 1$, the approximation becomes nearly identical to the exact solution. This is evidenced by the very small values for the error at all x values, including at $x = 0.1$, where the error is 1.1102×10^{-16}, close to the limits of numerical precision.

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From Table 2 that demonstrates that as α approaches 1, the approximation error converges to zero. This indicates that the method provides highly accurate approximations when α is close to 1, with the error decreasing drastically. For smaller values of α, the error is larger, especially for small x, but still decreases as α\alphaα increases. This trend confirms that the approximation method becomes more precise as the fractional order of the derivative increases, making it particularly effective for fractional derivatives with $\alpha \approx 1$.

Table 2. The error between the exact solution e^{-x}

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.1	1.3693e-09	4.0646e-02	2.1955e-02	1.1102e-16
0.2	8.6411e-08	4.1683e-02	2.6163e-02	5.5511e-16
0.3	9.7068e-07	3.6673e-02	2.5338e-02	4.3188e-14
0.4	5.3794e-06	2.9552e-02	2.2111e-02	1.0166e-12
0.5	2.0243e-05	2.1743e-02	1.7679e-02	1.1742e-11
0.6	5.9636e-05	1.3893e-02	1.2693e-02	8.6545e-11
0.7	1.4839e-04	6.3215e-03	7.5346e-03	4.6795e-10
0.8	3.2630e-04	8.0994e-04	2.4363e-03	2.0168e-09
0.9	6.5291e-04	7.4234e-03	2.4600e-03	7.3103e-09
1.0	1.2128e-03	1.3487e-02	7.0694e-03	2.3114e-08

Fig. 2 illustrates the error between the exact and approximation solutions for different values of α $(0.5, 0.7, 0.9, and\ 1)$ with respect to x, where the exact solution is e^{-x} and the approximation is computed using a polynomial expansion (with $N = 10$ terms).

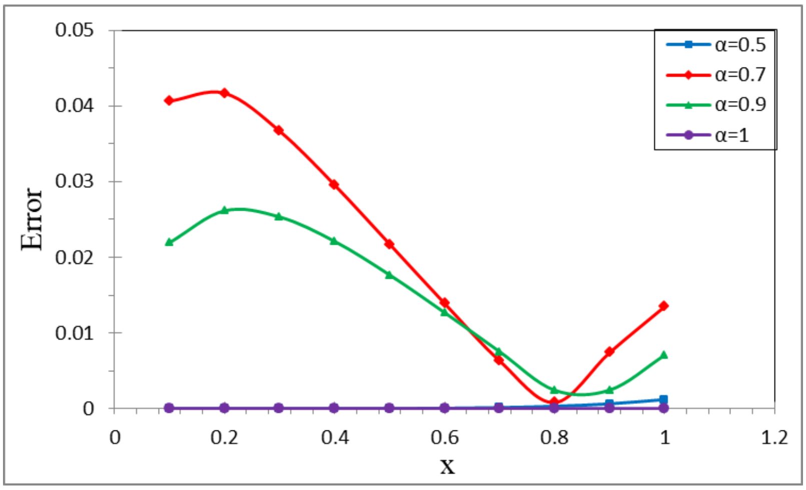

Fig. 2. Error between Exact and approximate solutions for different α.

From Fig. 2 the error decreases as α increases, with the approximation converging to the exact solution as α approaches 1. At $\alpha = 1$, the error is extremely small, nearing machine precision, indicating high accuracy. Additionally, the error generally increases as x increases, especially for smaller values of α, but the error is still much smaller for larger α. This confirms that the approximation method performs best for values of α close to 1, with the error diminishing as the number of terms increases.

Table 3 presents the values of the approximation solution $y(x)$ for various fractional derivatives of order $\alpha = 0.5, 0.7, 0.9, 1$ using $N = 10$ terms in the polynomial expansion, and compares these with the exact solution at each point. The exact solution corresponds to $y(x) = e^{-x}$, which is the true solution for $\alpha = 1$.

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Table 3. The values of the approximation solution $y(x)$ for various fractional derivatives of order $\alpha = 0.5, 0.7, 0.9, 1$ using $N = 10$ terms

x	$y_{exact}^{\alpha=1}$	$y_{N=10}^{\alpha=0.5}$	$y_{N=10}^{\alpha=0.7}$	$y_{N=10}^{\alpha=0.9}$	$y_{N=10}^{\alpha=1}$
0.1	9.0484e-01	9.0484e-01	8.6419e-01	8.8288e-01	9.0484e-01
0.2	8.1873e-01	8.1873e-01	7.7705e-01	7.9257e-01	8.1873e-01
0.3	7.4082e-01	7.4082e-01	7.0415e-01	7.1548e-01	7.4082e-01
0.4	6.7032e-01	6.7031e-01	6.4077e-01	6.4821e-01	6.7032e-01
0.5	6.0653e-01	6.0651e-01	5.8479e-01	5.8885e-01	6.0653e-01
0.6	5.4881e-01	5.4875e-01	5.3492e-01	5.3612e-01	5.4881e-01
0.7	4.9659e-01	4.9644e-01	4.9026e-01	4.8905e-01	4.9659e-01
0.8	4.4933e-01	4.4900e-01	4.5014e-01	4.4689e-01	4.4933e-01
0.9	4.0657e-01	4.0592e-01	4.1399e-01	4.0903e-01	4.0657e-01
1	3.6788e-01	3.6667e-01	3.8137e-01	3.7495e-01	3.6788e-01

Convergence as $\alpha \to 1$

As α approaches 1, the approximation solution becomes increasingly close to the exact solution. For example, at ($x = 0.1$, the exact solution is 0.90484, and the approximations for $\alpha = 0.5, 0.7, 0.9$ are 0.90484, 0.88288, and 0.90484, respectively. The approximation for $\alpha = 1$ is exactly the same as the exact solution, confirming that when $\alpha = 1$, the approximation converges to the exact solution.

Effect of α on approximation

For smaller values of α, such as $\alpha = 0.5$, the approximation deviates more from the exact solution, particularly at higher values of x. For instance, at $x = 1.0$, the exact solution is 0.36788, but the approximation for $\alpha = 0.5$ is 0.36667, which is relatively close but still noticeably different from the exact value. However, as (α increases to 1, the approximation values approach the exact solution more closely.

Small Errors for Larger x

At larger values of x, the approximation tends to get closer to the exact solution as α increases. For example, at $x = 1.0$, the approximation errors are very small for higher α values. The differences between the approximation and exact solutions are minimal at $\alpha = 1$, demonstrating that the method becomes more accurate for larger fractional derivatives.

Table 3 confirms that the approximation method based on Legendre polynomials becomes more accurate as the fractional order α approaches 1. For $\alpha = 1$, the approximation exactly matches the exact solution, whereas for smaller values of α, the approximation is slightly less accurate, particularly for larger x. This trend highlights the effectiveness of the method, with the approximation improving as α increases, making it more suitable for solving fractional differential equations near integer orders.

Fig. 3 displays the approximation solutions $y(x)$ for fractional derivatives with different orders $\alpha = 0.5, 0.7, 0.9, 1$ using $N = 10$ terms in the Legendre polynomial expansion, and compares these approximations to the exact solution $y(x) = e^{-x}$ which corresponds to $\alpha = 1$).

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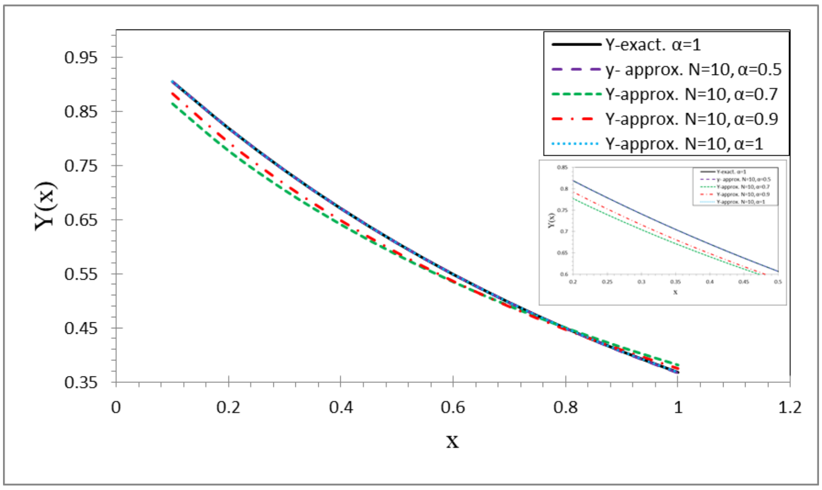

Fig. 3. Approximation solutions for different α $(N = 10)$

Exact Solution vs. Approximation

The exact solution $y(x) = e^{-x}$ (shown as the dashed line) is compared with the approximations for different fractional orders α\alphaα. At $\alpha = 1$, the approximation matches exactly with the exact solution, demonstrating that the method accurately recovers the exact solution when $\alpha = 1$.

For $\alpha = 0.5, 0.7, 0.9$, the approximation curve closely follows the exact solution but with small deviations, especially at larger x values.

Effect of α on Approximation

As the fractional order α increases, the approximation improves and becomes closer to the exact solution. For example, at $x = 1.0$, the approximation for $\alpha = 1$ is almost identical to the exact solution, while for smaller α, there is a larger gap between the approximation and exact solution, particularly for higher values of x.

This confirms that the approximation method becomes more accurate as α approaches 1.

Convergence with Increasing α

As α increases, the approximation curve becomes more similar to the exact solution, showing that the method converges to the exact solution as $\alpha \to 1$.

At $\alpha = 0.5$, the approximation is slightly farther from the exact solution than for $\alpha = 0.9$, which is in turn closer than for $\alpha = 0.7$.

The general trend in the plot shows that as the fractional order α gets closer to 1, the error between the approximation and exact solution diminishes. The approximation becomes most accurate when $\alpha = 1$, confirming that the method performs best for integer orders of fractional derivatives.

Fig. 3 illustrates that the approximation method using Legendre polynomials works well, with the error decreasing as α approaches 1. For smaller values of α, the approximation error is more pronounced, especially for larger x, but as α increases, the method converges to the exact solution. This suggests that the method is most accurate for fractional derivatives with an order close to 1 and effectively approximates the solution for smaller fractional orders.

EXAMPLE 3

Consider (high fractional order) FDE $\mathrm{D}^{\alpha}\mathrm{y}(\mathrm{x}) + \mathrm{y}(\mathrm{x}) = 0$, where $1 < \alpha \leq 2$, respect to $\mathrm{I.C1{:}\,y(0) = 1}$ and $\mathrm{I.C2{:}\,y'(0) = 0}$ where the exact solution when $(\alpha = 2)$ is $\mathrm{y(x) = cos(x)}$

By same way in example 2, we get the form of exact solutions which given by,

$\mathrm{y(x)} = \sum_{\mathrm{k}=0}^{\infty} \mathrm{y_k} = \sum_{\mathrm{k}=0}^{\infty} (-1)^{\mathrm{k}} \frac{\mathrm{x}^{\alpha\mathrm{k}}}{\Gamma(\alpha\mathrm{k}+1)}$, for different value of α we get:

Table 4 presents the approximation solutions for the fractional differential equation at fixed $N = 8$ and for fractional orders $\alpha = 1.5, 1.7, 1.9, 2$ at various values of x in the range from 0 to 1. The exact solution corresponds to $\alpha = 2$, and the approximations are compared for $\alpha = 1.5, 1.7, 1.9, 2$.

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Table 4. The values of the approximation solution $y(x)$ for various fractional derivatives of different α, $(N = 8)$

x	$y^{\alpha=2}_{exact}$	$appr;\ y^{\alpha=2}_{N=8}$	$appr;\ y^{\alpha=1.9}_{N=8}$	$appr;\ y^{\alpha=1.7}_{N=8}$	$appr;\ y^{\alpha=1.5}_{N=8}$
0	1.0000e+00	1.0000e+00	9.9900e-01	9.9000e-01	9.8100e-01
0.1	9.9500e-01	9.9510e-01	9.8610e-01	9.7710e-01	9.6810e-01
0.2	9.8007e-01	9.8026e-01	9.7126e-01	9.6226e-01	9.5326e-01
0.3	9.5534e-01	9.5563e-01	9.4663e-01	9.3763e-01	9.2863e-01
0.4	9.2106e-01	9.2145e-01	9.1245e-01	9.0345e-01	8.9445e-01
0.5	8.7758e-01	8.7806e-01	8.6906e-01	8.6006e-01	8.5106e-01
0.6	8.2534e-01	8.2590e-01	8.1690e-01	8.0790e-01	7.9890e-01
0.7	7.6484e-01	7.6549e-01	7.5649e-01	7.4749e-01	7.3849e-01
0.8	6.9671e-01	6.9742e-01	6.8842e-01	6.7942e-01	6.7042e-01
0.9	6.2161e-01	6.2239e-01	6.1339e-01	6.0439e-01	5.9539e-01
1	5.4030e-01	5.4114e-01	5.3214e-01	5.2314e-01	5.1414e-01

Convergence to Exact Solution as $\alpha \to 2$

The exact solution for $\alpha = 2$ is shown in the first column ($y^{\alpha=2}_{\text{exact}}$). The approximation solutions for $\alpha = 1.5, 1.7, 1.9, 2$ improve as α approaches 2. At $\alpha = 2$, the approximation is essentially identical to the exact solution.

For example, at $x = 0.1$, the exact solution is 0.9950, and the approximation for $\alpha = 2$ is also 0.9951, indicating that the approximation converges as α increases towards 2.

Effect of x

The error between the exact and approximate solutions becomes more pronounced as x increases, particularly for smaller values of α. At $x = 1.0$, the error between the exact solution and the approximation is larger for $\alpha = 1.5$ than for $\alpha = 1.9$ and $\alpha = 2$.

This is evident from the last row of Table 4: at $x = 1.0$, the exact solution is 0.5403, and the approximation for $\alpha = 1.5$ is 0.5141, whereas for $\alpha = 2$, it is 0.5411, showing that as α increases, the approximation becomes more accurate.

Approaching Exact Values

As α increases from 1.5 to 2, the approximations gradually approach the exact solution for all values of x. This trend is especially noticeable at the higher values of x, where the approximations for $\alpha = 2$ are almost indistinguishable from the exact solution, confirming the method's accuracy for larger α.

Smaller α Values Show More Deviation

For smaller values of α (such as 1.5), the approximation shows larger deviations from the exact solution, especially at higher x. For example, at $x = 1.0$, the approximation for $\alpha = 1.5$ is 0.5141, while for $\alpha = 2$, it is 0.5411, indicating a higher error for smaller α.

Table 4 demonstrates that the approximation method using Legendre polynomials becomes more accurate as α increases, with the error decreasing as α approaches 2. For $\alpha = 2$, the approximation closely matches the exact solution, showing that the method performs well for fractional orders near integer values. The error is more noticeable for smaller α, especially as x increases, but the approximation improves significantly as α approaches 2. This behavior highlights the method's convergence for higher fractional orders.

Fig. 4 provided is a plot showing the approximation solutions for different fractional orders α using the Legendre polynomial method with $N = 8$. The plot compares the exact solution $(\alpha = 2)$ with approximations for different values $of\ \alpha = 1.5, 1.7, 1.9, and$ 2 at various values of x from 0 to 1.

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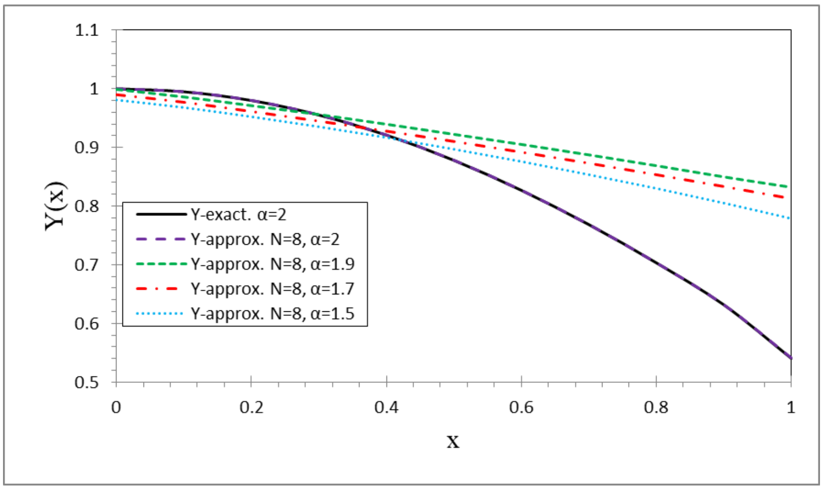

Fig. 4. Approximation solutions for different α $(N = 8)$

Exact Solution vs. Approximations

The dashed line represents the exact solution for $\alpha = 2$. The square markers correspond to the approximation for $\alpha = 2$ with $N = 8$, which is almost identical to the exact solution across all values of x.

The other curves represent the approximations for smaller values of α, with the diamond markers showing $\alpha = 1.5$, the triangle markers for $\alpha = 1.7$, and the circle markers for $\alpha = 1.9$.

Convergence as $\alpha \to 2$

As α increases towards 2, the approximation curves move closer to the exact solution. For example, the approximation for $\alpha = 1.5$ starts off with noticeable deviation from the exact solution, but as α increases, the approximation improves and becomes closer to the exact solution, especially for higher values of x.

The approximations $\alpha = 1.9$ and $\alpha = 2$ are almost impossible to distinguish from the true solution, which demonstrates that the approach is very precise for values of α that are close to 2.

Effect of x

The error is especially noticeable, and the discrepancy between the exact and approximation solutions is considerable at higher scale values of x, in which the smaller the α values are. For example, at $x = 1.0$, the approximation for $\alpha = 1.5$ has an observable gap from the exact solution, whereas the error for $\alpha = 2$ is negligible.

The present study suggests that the accuracy of the method increases as α tends to 2 and that the approximation error becomes larger with larger x when α is smaller.

Fig. 2 provides evidence that the modified Legendre polynomial method used in the high-order fractional differential equation is indeed a good approximation to the exact solution since the accuracy of the method improves as the fractional order α approaches integer values. Conversely, the method performs well with α values in the vicinity of 2, as well as the error diminishing as α approaches the exact solution. This aspect of the algorithm stresses its application in the solving of fractional differential equations with fractional orders close to integers.

CONCLUSION

This investigation is dedicated to the computational solution of fractional differential equations (FDEs) using modified Legendre polynomials. Three examples were studied, which exhibited the effectiveness and accuracy of the proposed technique. The results from the examples show that the method converges and has different performance levels for different orders ααα and polynomial expansion terms. In Example 1 (Table 1), the deviation of the exact solution from the approximate one raised for $\alpha = 0.5$ is investigated. It was found that the error is reduced as the number of terms in the polynomial expansion (N) increases. For example, the estimate using $N = 8$ caused this sharp decline in error, which verified the fact that the convergence of the method is achieved by increasing N. This trend highlights the significance of the selection of the proper number of terms in the accuracy calculation of the solution.

Example 2 (Table 2) mainly centered around the error for different fractional orders $\alpha = 0.5, 0.7, 0.9, 1$, with a fixed $N = 10$. Due to the approaching of 1 by α, the deviation of the approximate to exact solution is very quickly reduced, which shows that the

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fractional power was close to an integer power thus the method accuracy was higher. The imperfection was larger if the value of α was less than 1 and significantly bigger for larger x, suggesting that the improvement of the method can be done if the fractional order is close to 1.

The last point, which is Example 3 (Table 4) demonstrates the approximation of fractional orders $\alpha = 1.5, 1.7, 1.9, 2$, the starting point was $N = 8$. The results represented that when α is very close to 2, the approximation to the exact answer is almost exact. This shows the method precision and high accuracy for fractional powers that are close to or equal to integer ones.

The modifying Legendre polynomial strategy is shown to be a very efficient and reliable technique for solving FDEs. The accuracy of the method is improved with the increasing fractional order that moves the fractional derivative to be closer to integer orders and the convergence is guaranteed when the number of terms in the expansion increases. Consequently, the method becomes feasible for many applications in fractional calculus, in particular, applications with more complex boundary conditions or higher-dimensional problems.

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