Journal of Positive Sciences (JPS) eISSN: 2582-9351, https://positive-sciences.com JPS. 2024 Nov; 4(6): 124-139

Article info
Received on: 02.10.2024
Accepted on: 29.11.2024
Published on: 30.11.2024
doi: https://doi.org/10.52688/ASP46319

Research Article

Numerical solution of fractional differential equations using the Bernstein matrix for fractional differentiation

Ahmed M. Shukur [1, *]

[1] Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq

* ahmed.m.shokr@uotechnology.edu.iq

ABSTRACT

Fractional differential equations (FDEs) represent a highly significant aspect of physical and engineering models, albeit the exact solutions are still a difficult issue. In this research, we suggest using Bernstein polynomials to come up with numerical solutions to FDEs. The approach is based on the Bernstein operational matrix for fractional derivatives, which enables the FDE to be restated as a system of algebraic equations, and the system can be solved to find the unknown coefficients of approximation.
In addition to presenting the efficacy of the suggested method, we also utilize it for solving three different FDEs. As the first instance, a problem of a Bagley-Torvik equation is studied, and the result of $y(x) = 1 + x$, which is the exact solution, is achieved by the numerical method. The second example is concerned with the investigation of a linear fractional differential equation with boundaries, where the error is considerably decreased by increasing the degree of the polynomial n. Specifically, for $n = 3$, the L_2 error is 6.5×10^{-4}, and the L_∞ error is 1.5×10^{-3}, whereas for $n = 15$, these errors reduce to 4.6×10^{-7} and 6.7×10^{-7}, respectively. Through Example 3, you are faced with a scenario where the solution of the equation contains a square root. The results show that for $n = 3$, the L_2 error is 2.9×10^{-3} and L_∞ error is 6.5×10^{-3}, decreasing to 3.4×10^{-5} and 4.1×10^{-7}, respectively, for $n = 9$.
The developed technique offers a methodological, practical, and correct way to tackle FDEs. The computational results ascertain that the Bernstein polynomial method is a viable choice in solving FDEs very accurately.
Keywords: Numerical examples, Differential evolution algorithm, Bernstein matrix, Bagley-Torvik equation, errors

INTRODUCTION

Fractional differential equations (FDEs) have been increasingly worthy of researchers' attention for their ability to capture with high accuracy the memory and hereditary properties of complex physical, biological, and engineering systems [1]. While conventional differential equations are based on the concept of derivatives with whole numbers, FDEs are based on the idea of fractional derivatives, thus making the accuracy of the description of various real-world phenomena, such as viscoelastic materials, anomalous diffusion, control systems, possible [2].
A major challenge in solving FDEs is the intricacy of the equations which often makes it difficult to derive analytical solutions [3]. This is the reason why the numerical methods have been developed extensively to have the approaches to approximate the solutions with the highest accuracy [4]. Among these methods, polynomial-based approaches have been found to be extremely efficient [5]. Especially, the Bernstein polynomial is prominent for its fantastic approximation properties and easiness of use in numerical computations [6]. The Bernstein polynomials were introduced by the Russian mathematician S. N. Bernstein in 1912 and are a set of polynomials forming a complete basis for continuous functions over the interval [0, 1] [7]. They are widely used in numerical analysis, approximation theory, and computer-aided geometric design. The Bernstein polynomials of degree n are defined as [8]

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ k = 0, 1, \dots, n \tag{1}$$

where $B_{k,n}(x)$ is the binomial coefficient.
So by using Bernstein polynomials as the basis functions, one can devise (split) the numerical schemes that approximate the solutions of FDEs [9]. The technique adopted to conduct the study is the fractional differential method, allowing us not only to transfer the equations into the algebraic form but also to solve them numerically [10]. The fractional differential matrix is a matrix formed through the operational matrices of fractional derivatives from Bernstein polynomials [11]. These operational matrices make possible the transfer not only of the fractional derivatives but also of the vectors to the matrix-vector operations, thus significantly simplifying the calculation process. For a function f(x), the fractional derivative of order in the Caputo sense is [12]:

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_o^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \quad n-1 < \alpha < n \tag{2}$$

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

A fractional differentiation operation matrix can be applied on the function f(x) to express it as a linear combination of Bernstein polynomials leading to a system of algebraic equations that can be efficiently solved to find the coefficients [13].
Using Bernstein Polynomials to solve ordinary differential equations is a topic that has been extensively researched due to its properties of accuracy and stability [14]. The procedure of this method enables us to convert the differential equations to the algebraic equations, and then we can solve them efficiently by using matrix methods. Fractional differential equations are much more complex than ordinary ones due to their non-local and self-similarity, and thus additional difficulties are being raised. In the light of this, the spline method is used as a numerical tool which is effective. Particularly, the fractional-order spline provides a smooth way that is also flexible to deal with the approximation of solutions for fractional equations thus can produce a high degree of accuracy when capturing the fractional features of fractional systems [15].
Even though it is widely known how beneficial Bernstein polynomials are for solving classical differential equations, people are unaware of their application in dealing with fractional and fractional differential equations. And the fact is that the whole complexity of the fractional differential equations remained unexplored by the spline techniques so far, where the splines are well known for the approximation of a function. This very research aims to combine midpoints and fractional order splines to solve problems at the cross-cutting points of the two methods [16].
This research is an extension of the already existing numerical methods, which uses polynomial approximation techniques to address the issue of differential equations, by reducing equations to be algebraically solvable. In common with previous research, it employs the classical Bernstein polynomials to approximate the solutions of the differential equations [17].
The novel feature of this research is that it combines the Bernstein polynomial method with the spline method to obtain ultimately higher accuracy, mainly in the case of fractional differential equations. Also, fractional differentiation matrices that were built based on Bernstein polynomials will be addressed to optimize the computational efficiency.
One aim of this study is to examine the integration of adaptive mesh refinement techniques that significantly enhance the numerical accuracy of the proposed method. Moreover, the performance of the approach is endorsed by error analysis and convergence studies conducted as part of the verification process.
Although the method developed exhibits good progress, its handling of high dimensional problems and other issues such as the boundary condition requires more effort. To be specific, the research will take advantage of new techniques for the parallel processing and also move further in developing multi-dimensional fractional systems.
Among the obstcles towards efficient solution of FDEs are the computational costs, the stability, and the accuracy of numerical approximations. The research work carried out in this paper gives the field progress and stimulates it by providing a method that can be trusted and one that is time-saving while incorporating new ways such as fractional-order splines and differentiation matrices.
Fractional differential equations remain a challenging area of physics and mathematics due to the non-local nature of the operators and the complex structure of the operators. Many of the existing ways of anlaytic and/or simulation of the mathematical models are somewhat limited in the sense that they are only useful under certain circumstances; thus there is still a big gap for the development of efficient numerical techniques. To this end, a need for numerically accurate and computationally-efficient methods has paved the way for the application of Bernstein polynomials and spline techniques for solving such equations [18].
The focus of the study is to devise a highly efficient numerical method with the help of Bernstein Polynomials and fractional differentiation matrices to tackle and solve fractional and fractional differential equations.
Dealing with the Bernstein-based fractional differential matrix, the paper presents a quick numerical approach to the solution of FDEs. The suggested method not only furnishes highly precise approximations but is also of low computational complexity, thus it is also suitable for solving highly complex fractional-order systems. To be more specific, the paper is laid out as follows: In the next section, we will discuss in detail the fundamental of the Bernstein polynomials and their properties that are going to be used. The third section includes the fractional differential matrix and its construction in addition to an explanation. The numerical method and its practical part are outlined in the fourth section. The results of computational experiments, which are supported by suitable examples, are given in the fifth section, and finally, section 6 is devoted to the conclusion of the paper, followed by some suggestions for future research.

MATHEMATICAL PRELIMINARIES

Bernstein polynomials play a crucial role in numerical analysis, approximation theory, and fractional calculus. The given properties highlight their fundamental characteristics, particularly their role in fractional differential equations.

SOME PROPERTIES OF BERNSTEIN POLYNOMIALS [19]

a) Boundary Conditions

$B_{i,n}(1) = \delta_{in}$ and $B_{i,n}(0) = \delta_{i0}$ (3)

These properties indicate that Bernstein polynomials behave like Kronecker delta functions at the boundaries. When $x = 1$, only the term $i = n$ is nonzero, and when $x = 0$, only $i = 0$ is nonzero.

b) Symmetry Property

$B_{i,n}(1 - x) = B_{n-1,n}(x)$ (4)

This property shows the symmetry of Bernstein polynomials, meaning that evaluating a polynomial at $1 - x$ is equivalent to flipping the index.

c) Recurrence Relation

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x) \quad (5)$$

This recurrence relation allows for constructing Bernstein polynomials of order n using those of order $n-1$. It highlights their recursive nature, making them useful for numerical computations.

d) Linear Combination Property

$$B_{i,n-1}(x) = \left(\frac{n-i}{n}\right) B_{i,n}(x) + \left(\frac{i+1}{n}\right) B_{i+1,n}(x) \quad (6)$$

The given identity manifests the connection of neighboring Bernstein polynomials and offers a different method of calculating them.

e) Non-negativity Property

$$B_{i,n}(x) \geq 0, \quad \forall x \in [0,1] \quad (7)$$

This provides the assurance that Bernstein polynomials will never take on negative values, a property of much use in probability theory and in the approximation theory.

f) Partition of Unity Property

$$\sum_{i=0}^{n} B_{i,n}(x) = 1 \quad (8)$$

This clearly indicates that Bernstein polynomials are a partition of unity, which is the very foundation of function approximation.

g) Differentiation Formula (Fractional Form)

$$\frac{d}{dx}\left(B_{i,n}(x)\right) = \frac{i-nx}{x(1-x)} B_{i,n}(x) \quad (9)$$

This one offers a formula that can be used to find the derivatives of Bernstein polynomials.

h) Differentiation Formula (Recursive Form)

$$\frac{d}{dx}\left(B_{i,n}(x)\right) = n\left[B_{i-1,n-1}(x) - B_{i,n-1}(x)\right] \quad (10)$$

This other representation of the derivative allows us to express them using Bernstein polynomials that are of a lower order.

i) Integral Property

$$\int_0^1 B_{i,n}(x)\,dx = \frac{1}{n+1} \quad (11)$$

The integrality feature of this property indicates that the average of the Bernstein polynomial over the interval [0,1] is smooth.

j) Multiplication Property

$$B_{i,n}(x)B_{j,n}(x) = \frac{\binom{n}{i}\binom{n}{j}}{\binom{n+m}{i+j}} B_{i+j,n+m}(x) \quad (12)$$

That property tells us that the product of two Bernstein polynomials can be expressed in a higher-order Bernstein polynomial. Such properties become very important when solving fractional differential equations numerically with the help of Bernstein matrices, as in that way it is possible to approximate functions, compute derivatives, and integrate solutions in an orderly manner.

APPROXIMATION OF A FUNCTION BY A BERNSTEIN POLYNOMIAL

A Hilbert space $H = L^2[0,1)$ has been supposed with scalar product as [20]

$$\langle f,g\rangle = \int_0^1 f(x)g(x)dx \quad (13)$$

and the corresponding norm

$$\|f\|_2 = \sqrt{\langle f,f\rangle} \quad (14)$$

Let $\{b_0^n(x), b_1^n(x), \dots, b_n^n(x)\} \subset H$, be the set of Bernstein polynomials of degree n. The space spanned by these polynomials is:

$$S_n = \mathrm{Span}\{b_0^n(x), b_1^n(x), \dots, b_n^n(x)\} \quad (15)$$

Since S_n is a finite-dimensional vector subspace, there exists a unique best approximation for any function $f(x) \in L^2[0,1)$ such that

$$\forall z \in S_n, \exists!\, f_n \in S_n : \|f - f_n\| \leq \|f - z\| \quad (16)$$

Since $f_n \in S_n$, it can be represented as:

$$f_n(x) = \sum_{i=0}^{n} C_i b_i^n(x) = C^T B(x), \quad (17)$$

where $B(x)$ is the Bernstein basis vector

$$B(x) = [b_0^n(x), b_1^n(x), \dots, b_n^n(x)]^T \quad (18)$$

and C^T is the coefficient vector:

$$C^T = [C_0, C_1 \dots, C_n] \quad (19)$$

The coefficients C^T can be computed using the orthogonality condition

$$C^T\langle B(x), B(x)\rangle = \langle f, B(x)\rangle \quad (20)$$

where

$$\langle f, B(x)\rangle = \int_0^1 f(x)B(x)^T dx = [\langle f, b_0^n\rangle, \langle f, b_1^n\rangle, \dots, \langle f, b_n^n\rangle], \quad (21)$$

The matrix Q is defined as

$$Q = \langle B(x), B(x)\rangle = \int_0^1 B(x)B(x)^T dx \quad (22)$$

and the coefficients are given by

$$C^T = \left(\int_0^1 f(x)B(x)^T dx\right) Q^{-1} \quad (23)$$

This formulation allows us to compute the best approximation of any function using Bernstein polynomials.

ANALYTIC OF CONVERGENCE OF BERNSTEIN POLYNOMIALS [21]

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

Let $f \in C^{(n+1)}[0,1)$ be a sufficiently smooth function and
$S_n = \mathrm{Span}\{b_0^n(x), b_1^n(x), \ldots, b_n^n(x)\}$ denote the space spanned by Bernstein polynomials of degree n.
Suppose that $C^T B(x)$ is the best approximation for $f(x)$ in S_n, then we have the following bound for the approximation error

$$\|f - C^T B(x)\|_{L^2[0,1)} \le \frac{\gamma}{(n+1)!\sqrt{2n+3}} \quad (24)$$

where

$$\gamma = \max_{x\in[0,1)} \left|f^{(n+1)}(x)\right| \quad (25)$$

This inequality guarantees that the approximation error decreases as n increases, providing theoretical justification for the convergence of the Bernstein approximation.

PROOF OF CONVERGENCE ESTIMATE

Expanding $f(x)$ using Taylor series at $x = 0$, we obtain

$$f_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) \quad (26)$$

Using the Taylor remainder theorem, we estimate the error

$$|f(x) - f_n(x)| \le \frac{x^{n+1}}{(n+1)!}\left|f^{(n+1)}(\xi)\right| \text{, for some } \xi \in [0,1], \quad (27)$$

Since $C^T B(x)$ is the best approximation for $f(x)$ in S_n and $f_n \in S_n$, the by using we use the above inequality

$$\|f - C^T B(x)\|^2_{L^2[0,1)} \le \|f - f_n(x)\|^2_{L^2[0,1)} \quad (28)$$

Computing the integral norm

$$\int_0^1 |f(x) - f_n(x)|^2 dx = \int_0^1 \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi) dx \quad (29)$$

Simplifying

$$\int_0^1 \left(\frac{x^{n+1}}{(n+1)!}\right)^2 \left(f^{(n+1)}(\xi)\right)^2 dx \quad (30)$$

Since $\gamma = \max_{x\in[0,1]}\left|f^{(n+1)}(x)\right|$, we take the supremum over ξ

$$\le \frac{\gamma^2}{\left((n+1)!\right)^2}\int_0^1 x^{2n+2} dx \quad (31)$$

Evaluating the integral:

$$\int_0^1 x^{2n+2} dx = \frac{1}{2n+3} \quad (32)$$

Thus, we obtain:

$$\|f - f_n(x)\|^2_{L^2[0,1)} \le \frac{\gamma^2}{\left((n+1)!\right)^2(2n+3)} \quad (33)$$

Taking the square root:

$$\|f - C^T B(x)\|_{L^2[0,1)} \le \frac{\gamma}{(n+1)!\sqrt{2n+3}} \quad (34)$$

- This result provides a quantitative measure of how well Bernstein polynomials approximate smooth functions.
- The bound suggests that the approximation error diminishes rapidly as n increases, particularly for functions with smooth higher-order derivatives.
- The factorial term $(n + 1)!$ in the denominator confirms that for sufficiently large n, the Bernstein polynomial approximation converges exponentially.
- If f is analytic, the error decreases even faster, leading to near-exact approximations for moderate n.
- In practice, the rate of convergence may depend on function regularity and boundary behavior.

BASIS AND DEFINITIONS: FUNDAMENTAL CONCEPTS FOR THIS STUDY

In this section, we present the fundamental properties of Bernstein polynomials, their matrix representation, and their role in function approximation.

DEFINITION OF BERNSTEIN POLYNOMIALS

The Bernstein polynomial of degree n on the interval $[0, 1]$ is defined as [22]:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad (35)$$

is the binomial coefficient.
An alternative definition for Bernstein polynomials over a general interval $[a, b]$ is:

$$b_i^n(x) = \binom{n}{i}\frac{(b-a)^i(b-x)^{n-i}}{(b-a)^n}, \; i = 0, 1, \ldots, n \quad (36)$$

where the polynomial vanishes for indices outside the valid range:

$$b_i^n(x) = 0, \text{ if } i < 0 \text{ or } i > n \quad (37)$$

For example, when $n = 5$, the Bernstein basis polynomials are:

$$b_0^5(x) = (1 - x)^5$$

$$b_1^5(x) = 5x(1 - x)^4$$

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

$b_2^5(x) = 10x^2(1-x)^3$
$b_3^5(x) = 10x^3(1-x)^2$
$b_4^5(x) = 5x^4(1-x)$
$b_5^5(x) = x^5$
Fig. 1 illustrates the above polynomials for $n = 5$

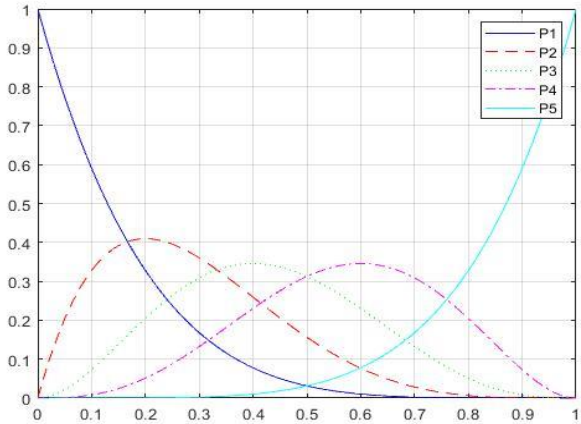

Fig. 1. Prinstain polynomial for $n = 5$

TRANSFORMATION OF BERNSTEIN POLYNOMIALS USING NEWTON'S BINOMIAL THEOREM

By applying Newton's binomial expansion to $(1-x)^{n-i}$, we express the Bernstein basis as a polynomial expansion [23]:

$$b_i^n(x) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{i} \binom{n-i}{j-i} x^i \ , \ i = 0, \dots, n \tag{38}$$

Rearranging, we obtain

$$b_i^n = A_{i+1} T_n(x), \ i = 0, \dots, n \tag{39}$$

where

A_{i+1} is a row vector of coefficients,

$$A_{i+1} = \left[\overbrace{0,0,\dots,0}^{i} (-1)^0 \binom{n}{i}, (-1)^1 \binom{n}{i} \binom{n-i}{1}, \dots, (-1)^{n-i} \binom{n}{i} \binom{n-i}{n-i} \right] \tag{40}$$

$T_n(x)$ is the vector of monomials,

$$T_n(x) = [1, x, \dots, x^n]^T \tag{41}$$

MATRIX REPRESENTATION OF BERNSTEIN BASIS

We define an $(n+1) \times (n+1)$ transformation matrix A whose rows correspond to the coefficient vectors $A_1, A_2, \dots, A_{n+1}$[24]

$$A = [A_1, A_2, \dots, A_{n+1}]^T \tag{42}$$

Then, the Bernstein basis vector can be written in matrix form as:

$$B(x) = AT_n(x) \tag{43}$$

where

$$B(x) = [b_0^n(x), b_1^n(x), \dots, b_n^n(x)]^T \tag{44}$$

Explicit Form of the Transformation Matrix A

The transformation matrix A is structured as follows:

$$A = \begin{bmatrix} (-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0} \binom{n-0}{1} & \cdots & (-1)^{n-0} \binom{n}{0} \binom{n-0}{n-0} \\ 0 & (-1)^0 \binom{n}{1} & \cdots & (-1)^{n-1} \binom{n}{1} \binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0 \binom{n}{n} \end{bmatrix}, \tag{45}$$

Since the determinant of A is given by:

$$\det(A) = \prod_{i=0}^{n} \binom{n}{i} \neq 0, \tag{46}$$

The matrix A is invertible, allowing us to express $T_n(x)$ in terms of Bernstein polynomials:

$$T_n(x) = A^{-1} B(x) \tag{47}$$

Discussion and Applications

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

- Linear Independence: The Bernstein basis functions form a linearly independent set over [0,1], making them a valid polynomial basis.
- Polynomial Approximation: Any smooth function on [0,1] can be approximated using the Bernstein polynomial expansion.
- Matrix Formulation: The transformation matrix A provides a systematic way to transition between monomial and Bernstein basis representations.
- Computational Efficiency: The explicit formula for A^{-1}allows for efficient numerical implementation of function approximations.

This section establishes the Bernstein polynomials' fundamental properties and their matrix representation. This foundation is critical for their application in solving fractional and fractional differential equations using the Bernstein operational matrix method.

FRACTIONAL DIFFERENTIAL EQUATIONS AND BERNSTEIN MATRIX APPROACH

THE GENERAL FORM OF FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional differential equations (FDEs) generalize classical differential equations by allowing differentiation of non-integer order. They arise in various fields, such as control theory, viscoelasticity, and anomalous diffusion [25].

A general linear fractional differential equation is given by:

$$D^{\alpha}y(x) = \sum_{i=1}^{k} a_i D^{\beta_i} y(x) + a_{k+1}y(x) + g(x) \quad (48)$$

where: D^{α} denotes the fractional derivative of order α, a_i $(i = 1, \dots, k+1)$ are real constant coefficients, $g(x)$ is a given function (forcing term), and the orders satisfy $n-1 < \alpha \le n$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$.

The initial conditions are given as:

$$y^{(i)}(0) = d_i, \quad i = 0,1 \dots, n-1 \quad (49)$$

where d_i are known constants.

FRACTIONAL DIFFERENTIATION OPERATORS

Fractional derivatives are defined using different approaches, with the two most common being:

RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE

The Riemann–Liouville fractional derivative of order α\alphaα is defined as [26]:

$$D^{\alpha}y(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}y(t)dt, \quad (50)$$

where $n = \lceil\alpha\rceil$ is the smallest integer greater than or equal to α, and $\Gamma(\cdot)$ is the Gamma function.

CAPUTO FRACTIONAL DERIVATIVE

The Caputo fractional derivative of order α\alphaα is given by [27]:

$$C_{D^{\alpha}}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}y^n(t)dt \quad (51)$$

The Caputo derivative is often preferred in physical applications because it allows for intuitive initial conditions involving integer-order derivatives.

FRACTIONAL DIFFERENTIAL EQUATIONS IN MATRIX FORM

To numerically approximate fractional derivatives, we use the Bernstein polynomial expansion. Suppose $y(x)$ can be approximated using Bernstein polynomials:

$$y(x) \approx \sum_{i=0}^{n} C_i b_i^n(x) \quad (52)$$

where C_i are the unknown coefficients, and $b_i^n(x)$ are the Bernstein basis functions.

Applying the Bernstein operational matrix for fractional differentiation D_B^{α}, we approximate the fractional derivative as:

$$D^{\alpha}y(x) \approx D_B^{\alpha}C \quad (53)$$

Substituting this into the given fractional differential equation:

$$D_B^{\alpha}C = \sum_{i=1}^{k} a_i D_B^{\beta}C + a_{k+1}C + g(x) \quad (54)$$

Rearranging, we obtain a system of algebraic equations:

$$\left(D_B^{\alpha} - \sum_{i=1}^{k} a_i D_B^{\beta} - a_{k+1}I\right)C = G, \quad (55)$$

where I is the identity matrix and G represents the transformed right-hand side function $g(x)$.

ADVANTAGES OF THE BERNSTEIN APPROACH FOR FDES

- Numerical Stability: The Bernstein polynomial basis ensures smooth approximations.
- Accuracy: The operational matrix for fractional differentiation provides precise approximations.

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

- Efficient Computation: The method converts the FDE into a system of algebraic equations, making it solvable using numerical techniques.

BERNSTEIN'S MATRIX FOR OPERATIONS OF FRACTIONAL DIFFERENTIATION

Let $D^{\alpha}B(x) \cong D^{(\alpha)}B(x)$ (56)

where $D^{(\alpha)}$ is $(n+1)$ by $(n+1)$ matrix of operations of fractional differential by Caputo with order with order α, which define as [28]:

$$D^{\alpha} = \begin{bmatrix} \sum_{j=\lceil\alpha\rceil}^{n}\omega_{0,j,0} & \sum_{j=\lceil\alpha\rceil}^{n}\omega_{0,j,1} & \cdots & \sum_{j=\lceil\alpha\rceil}^{n}\omega_{0,j,n} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,0} & \sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,1} & \cdots & \sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,n} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{j=\lceil\alpha\rceil}^{n}\omega_{n,j,0} & \sum_{j=\lceil\alpha\rceil}^{n}\omega_{n,j,1} & \cdots & \sum_{j=\lceil\alpha\rceil}^{n}\omega_{n,j,n} \end{bmatrix} \quad (57)$$

where $\omega_{i,j,l}$ given by the form:

$$\omega_{i,j,l} = (-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}\sum_{k=0}^{n}\lambda_{lk}\mu_{kj} \quad (58)$$

Also

$$\left.\begin{aligned} &\mu_{kj} \text{ given as: } \mu_{kj} = \sum_{s=k}^{n}(-1)^{s-k}\binom{n}{i}\binom{n-k}{s-k}\frac{1}{j-\alpha+s+1}, \qquad \text{and} \\ &\lambda_{lk} = \frac{(-1)^{j+k}}{\binom{n}{j}\binom{n}{k}}\sum_{i=0}^{\min(j,k)}(2i+1)\binom{n+i+1}{n-j}\binom{n-i}{n-j}\binom{n+i+1}{n-k}\binom{n-i}{n-k} \end{aligned}\right\} \quad (59)$$

Prove that: since the analytic form of Bernstein polynomial in (3) and the both Caputo fractional derivatives that given by (60-61):

$$D^{\alpha}x^{j} = \begin{cases} 0 & \text{, if } j \in \mathbb{N}\cup\{0\} \text{and } j < \lceil\alpha\rceil \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}x^{j-\alpha}, & \text{if } j \in \mathbb{N}\cup\{0\}, j \geq \lceil\alpha\rceil \text{OR } j \notin \mathbb{N}, j > \lfloor\alpha\rfloor \end{cases} \quad (60)$$

$$D^{\alpha}\big(\lambda f(x) + \mu g(x)\big) = \lambda D^{\alpha}f(x) + \mu D^{\alpha}g(x) \quad (61)$$

We can get,

$$D^{\alpha}b_i^n(x) = \sum_{j=\lceil\alpha\rceil}^{n}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}x^{j-\alpha} \quad (62)$$

We approximate $x^{j-\alpha}$ by Bernstein polynomial to get,

$$x^{j-\alpha} \cong \sum_{l=0}^{n}\mu_{lj}\, b_l^n(x) \quad (63)$$

where

$$\mu_{lj} = \int_0^1 x^{j-\alpha}d_l^n(x)dx \quad (64)$$

$$= \sum_{k=0}^{n}\lambda_{lk}\int_0^1 x^{j-\alpha}d_k^n(x)dx$$

$$= \sum_{k=0}^{n}\lambda_{lk}\sum_{s=k}^{n}(-1)^{s-k}\binom{n}{k}\binom{n-k}{s-k}\int_0^1 x^{j-\alpha+s}dx$$

$$= \sum_{k=0}^{n}\lambda_{lk}\sum_{s=k}^{n}(-1)^{s-k}\binom{n}{k}\binom{n-k}{s-k}\frac{1}{(j-\alpha+s+1)}$$

$$= \sum_{k=0}^{n}\lambda_{lk}\mu_{lj}, \quad (65)$$

and

$$d_j^n(x) = \sum_{k=0}^{n}\lambda_{jk}\, b_j^n(x), \; j = 0,1,\ldots,n \quad (66)$$

By using Eq. 62 and Eq. 63 we can get

$$D^{\alpha}b_i^n(x) \cong \sum_{j=\lceil\alpha\rceil}^{n}\sum_{l=0}^{n}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}\mu_{lj}b_j^n(x) \quad (67)$$

$$= \sum_{l=0}^{n}\left(\sum_{j=\lceil\alpha\rceil}^{n}(-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}\sum_{k=0}^{n}\lambda_{lk}\mu_{kj}\right)b_l^n$$

$$= \sum_{l=0}^{n}\left(\sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,l}\right)b_l^n(x) \quad (68)$$

where $\omega_{i,j,l}$ given in (58), now we can rewrite eq. (68) as a vector,

$$D^{\alpha}b_i^n(x) \cong \left[\sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,0}\,, \sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,1}\,, \ldots, \sum_{j=\lceil\alpha\rceil}^{n}\omega_{i,j,n}\right]B(x), i = 0,\ldots,n \quad (69)$$

this will be mean proof done.

NUMERICAL SOLUTION OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

Fractional differential equations (FDEs) arise in various applications, including physics, engineering, and biology, where memory effects and anomalous diffusion play a significant role. Solving such equations analytically is often challenging, necessitating the

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

development of efficient numerical techniques. The Bernstein polynomial method provides a systematic approach by transforming FDEs into algebraic systems that can be solved using matrix techniques.

CONVERSION FDES TO ALGEBRAIC EQUATIONS

When we use the Bernstein operational matrix to deal with the fractional differentiation, the solution of an FDE can be represented with Bernstein polynomials approximations. The form of the fractional differential equation can be written as the general case [29]:

$$F\left(x, y(x), D^{\beta_1}y(x), \dots, D^{\beta_k}y(x)\right) = 0 \quad (70)$$

subject to boundary conditions

$$H_i\left(y(\xi_i), y'(\xi_i), \dots, y^{(p)}(\xi_i)\right) = d_i, i = 0,1, \dots, p \quad (71)$$

where

$\xi_i \in [0, 1]$ are given boundary points, p statistics $0 \le p < \max\{\beta_i, i = 1, \dots, k\} \le p + 1$, H_i represents a set of linear operators acting on $y(x)$, F may be nonlinear in general cases.

APPROXIMATION USING BERNSTEIN POLYNOMIALS

The unknown function $y(x)$ is approximated as [30]:

$$y(x) \cong \sum_{i=0}^{n} c_i b_i^n(x) = C^T B(x) \quad (72)$$

where

$B(x) = [b_0^n(x), b_0^n(x), \dots, b_n^n(x)\ ]^T$is the vector of Bernstein basis functions,

$C = [C_0, C_1, \dots, C_n]^T$is the coefficient vector to be determined.

Utilizing the Bernstein operational matrix D_B^{β} for fractional differentiation:

$$D^{\beta_i}y(x) \approx D^{\beta_i}C \quad (73)$$

Thus, substituting into equation (70):

$$F\left(x, C^T B(x), C^T D^{(\beta_1)}B(x), \dots, C^T D^{(\beta_k)}B(x)\right) = 0 \quad (74)$$

The problem is reduced to a set of algebraic equations that are linear in nature:

$$AC = G, \quad (75)$$

where A is a matrix which is realized with the help of the Bernstein operational matrix and G is the transformation of the given function on the right-hand side. In the case of nonlinear FDEs, the methods that might be used are Newton's method and the Picard iteration technique to solve the system that has been obtained.

HANDLING BOUNDARY CONDITIONS

The boundary conditions (71) place an extra set of restrictions on the system. Approximating to the case [31]:

$$y(\xi_i) \approx C^T B(\xi_i). \quad (76)$$

and similarly for higher-order derivatives, we obtain a set of algebraic conditions:

$$H_i\left(C^T B(\xi_i), C^T D^{(1)}B(\xi_i), \dots, C^T D^{(p)}B(\xi_i)\right) = d_i\ , i = 0,1, \dots, p \quad (77)$$

Such constraints are merged with the algebraic system in a way that will not only enable a solution but will also satisfy the established boundary conditions.

ADVANTAGES OF THE BERNSTEIN APPROACH FOR FDES

- Accuracy: This method lets you find the appropriate results with a fixed level of accuracy.
- Efficiency: By using the operational matrix approach one transforms the FDE into a set of algebraic equations and thus it is computationally efficient.
- Flexibility: In its usage, it is acceptable with fractional differential equations of both linear and nonlinear nature.
- Applicability: Moreover, the technique can be applied to more local evolution problems and fractional differential equations.

NUMERICAL SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING BERNSTEIN POLYNOMIALS

For the numerical solution of the fractional differential equations (FDEs), we use the Bernstein polynomials to approximately solve and convert the problem to a system of algebraic equations.

Step 1: Approximate $y(x)$ Using Bernstein Polynomials

The function $y(x)$ is approximated as:

$$y(x) \approx \sum_0^n c_i\, b_i^n(x) = C^T B(x) \quad (78)$$

where

$$B(x) = [b_0^n(x), b_1^n(x), \dots, b_n^n(x)\ ]^T \text{ is the Bernstein basis vector,} \quad (79)$$

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

$C = [C_0, C_1, \dots, C_n]^T$ is the unknown coefficient vector to be determined.

Step 2: Use Bernstein Operational Matrix To Calculate Fractional Derivative

Utilizing a fractional differentiation Bernstein operational matrix, we assume the fractional derivatives:

$$D^{\beta_j} y(x) \cong C^T D^{\beta_j} B(x),\ j = 1, \dots, k \quad (80)$$

Substituting this into the given FDE:

$$F\left(x, C^T B(x), C^T D^{(\beta_1)} B(x), \dots, C^T D^{(\beta_k)} B(x)\right) = 0 \quad (81)$$

This equation must be satisfied at specific points in the domain.

Step 3: Apply Boundary Conditions

Similarly, substituting the approximation $y(x)$ into the boundary conditions:

$$H_i\left(C^T B(\xi_i), C^T D^{(1)} B(\xi_i), \dots, C^T D^{(p)} B(\xi_i)\right) = d_i\ , i = 0,1, \dots, p \quad (82)$$

This results in $(p + 1)$ algebraic equations.

Step 4: Choose Nodes for Collocation Method

To solve for the unknown coefficients C, we enforce equation (23) at specific nodes. The best choice for these nodes is the roots of the Chebyshev polynomials:

$$x_i = \left(\frac{1}{2}\right)\left(\cos\left(\frac{i\pi}{n}\right) + 1\right),\ \ i = 1, \dots, n - p \quad (83)$$

The utilization of these nodes results in the solution becoming more numerically stable and accurate: Consequently, we get a system of (n+1) algebraic equations from

- $(n - p)$ equations of the given fractional differential equation (at Chebyshev nodes)).
- $(p + 1)$ equations from the boundary conditions.

Step 5: Solve the System

n the system of $(n + 1)$ algebraic equations, the solution can be obtained through the common methods of numerical computing:

- Gaussian elimination,
- LU decomposition,
- Newton's method (if the system is nonlinear).

After solving for the coefficients c_i, the approximate solution $y(x)$ is obtained as:

$$y \approx \sum_{i=0}^{n} c_i b_i^n(x) \quad (84)$$

- Approximate $y(x)$ using Bernstein polynomials.
- Compute Bernstein operational matrices for fractional derivatives.
- Formulate a system of algebraic equations using collocation at Chebyshev nodes.
- Incorporate boundary conditions.
- Solve for the unknown coefficients c_i.
- Construct the approximate solution.

This method efficiently transforms FDEs into algebraic systems, making them easier to solve numerically.

NUMERICAL IMPLEMENTATION AND COMPUTATIONAL RESULTS

To demonstrate the efficiency and simplicity of the Bernstein polynomial method, we apply it to a well-known fractional differential equation.

EXAMPLE 1: BAGLEY-TORVIK NONHOMOGENEOUS DIFFERENTIAL EQUATION

We consider the fractional differential equation:

$$D^2 y(x) + D^{3/2} y(x) + y(x) = 1 + x \quad (85)$$

with initial conditions

$y(0) = 1\ , y'(0) = 1$

Using the Bernstein polynomial approximation for $m = 2$, we approximate the solution as

$$y(x) \cong c_0 b_0^2(x) + c_1 b_1^2(x) + c_2 b_2^2(x) = C^T B(x), \quad (86)$$

where

$B(x) = [b_0^2(x), b_1^2(x), b_2^2(x)]^T$

The corresponding Bernstein polynomials are

$b_0^2(x) = (1 - x)^2,$

$b_1^2(x) = 2x(1 - x),$

$b_2^2(x) = x^2$

Step 1: Compute the Bernstein Operational Matrices

We use the operational matrices for differentiation:

$$D^{(1)} = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}, \quad (87)$$

$$D^{(2)} = \begin{bmatrix} 2 & 2 & 2 \\ -4 & -4 & -4 \\ 2 & 2 & 2 \end{bmatrix}, \quad (88)$$

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

$$D^{\left(\frac{3}{2}\right)} = \left(\frac{8}{35\sqrt{\pi}}\right)\begin{bmatrix} 3 & 15 & 17 \\ -6 & -30 & -34 \\ 3 & 15 & 17 \end{bmatrix} \quad (89)$$

Step 2: Apply the Fractional Differential Equation

By substituting the approximated solution and its derivatives into the differential equation at the collocation point $x_1 = \frac{1}{2}$, we obtain:

$$\frac{9}{4}(c_0 + c_2) - \frac{7}{2}c_1 + \left(\frac{20}{7\sqrt{\pi}}\right)(c_0 - 2c_1 + c_2) - \frac{3}{2} = 0 \quad (90)$$

Step 3: Apply the Boundary Conditions

From $y(0) = 1$, we obtain

$$c_0 - 1 = 0 \quad (91)$$

From $\acute{y}(0) = 1$, we obtain

$$-2c_0 + 2c_2 - 1 = 0 \quad (92)$$

Step 4: Solve for c_0, c_1, c_2

Solving the system of equations

$$y(x) = \left(1, \frac{3}{2}, 2\right)\begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix} \quad (93)$$

Expanding

$$y(x) = 1(1 - 2x + x^2) + \frac{3}{2}(2x - 2x^2) + 2x^2 \quad (94)$$

$$= 1 - 2x + x^2 + 3x - 3x^2 + 2x^2$$

$$= 1 + x \quad (95)$$

Step 6: Compare with the Exact Solution

The exact solution to the differential equation is $y(x) = 1 + x$, which matches our numerical solution exactly. This confirms the accuracy and efficiency of the Bernstein polynomial approach.

Fig. 1 presents the comparing the exact solution $y(x) = 1 + x$ with the Bernstein polynomial approximation. The exact solution is shown as a dashed blue line, while the Bernstein approximation is represented by the solid red line. The two curves match perfectly, indicating that the method accurately approximates the solution.

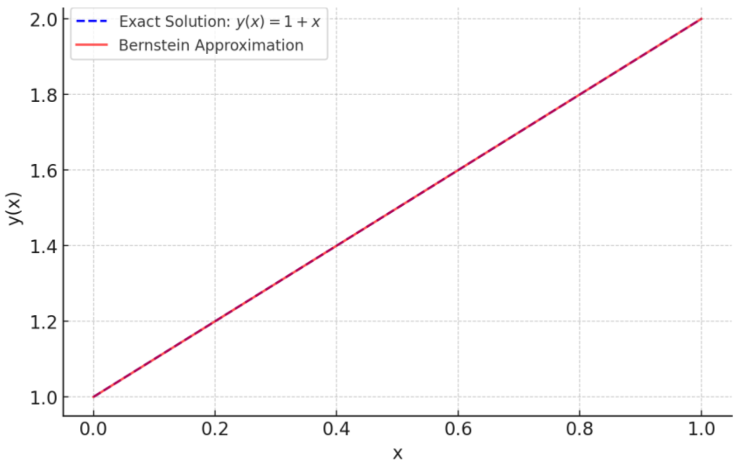

Fig. 1. Comparison of the exact solution and Bernstein polynomial approximation

From this example, We successfully used Bernstein polynomials to approximate the solution of a fractional differential equation. The method reduced the problem to a system of algebraic equations, which were easy to solve. The numerical solution matched the exact solution, showing the power of the Bernstein polynomial approach.

EXAMPLE 2

Consider the linear fractional order differential equation with boundary conditions,

$$4(x+1)D^{\frac{5}{2}}y(x) + 4D^{\frac{3}{2}}y(x) + \frac{1}{\sqrt{x+1}}y(x) = \sqrt{x} + \sqrt{\pi} \quad (96)$$

with boundary conditions

$$y(0) = \sqrt{\pi}, y'(0) = \frac{\sqrt{\pi}}{2}, y(1) = \sqrt{2\pi} \quad (97)$$

where the exact solution is given by

$$y(x) = \sqrt{\pi(x+1)} \quad (98)$$

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

Using the same methodology as an example 1 (applying Equations 70-77), we approximate y(x) using Bernstein polynomials for different values of n (3, 6, 12, 15). Table 1 presents the L_2 and L_∞ errors for each approximation, demonstrating the increasing accuracy of the method as n increases.
By using the same equations in (70 to 77) and same steps in example1, with different values of $(n = 3, 6, 12, 15)$, Table 1 shows the L_∞error and L_2error .

Table 1. Error analysis for different values of n

n	L_2 error	L_∞ error
3	6.5×10^{-4}	1.5×10^{-3}
6	6.1×10^{-6}	1.6×10^{-5}
12	9.7×10^{-7}	1.4×10^{-6}
15	4.6×10^{-7}	6.7×10^{-7}

Table 1 presents the L_2 and L_∞ errors for different values of n when solving the given fractional differential equation using the Bernstein polynomial approximation. The data reveals a clear trend: as n increases from 3 to 15, both error metrics significantly decrease.
The L_2 error, which represents the overall deviation of the approximate solution from the exact solution in terms of mean squared difference, reduces from 6.5×10^{-4} for $n = 3$ to 4.6×10^{-7} for $n = 15$.
From this Table
Influence of Increasing n on accuracy:

- When the count of Bernstein basis functions n goes up, both the L_2 and L_∞ errors decrease heavily.
- For $n = 3$, the L_2 error is 6.5×10^{-4}, which is relatively large, indicating a less accurate approximation.
- For $n = 15$, the L_2 error reduces to 4.6×10^{-7}, showing a substantial improvement in approximation accuracy.

1. Comparison Between L_2 and L_∞ Errors

- The L_∞ error is consistently larger than the L_2 error, which suggests that the maximum pointwise error at some specific locations (e.g., near boundaries) is higher than the average error across the interval.
- This behavior is expected because polynomial approximations can exhibit slight oscillations, leading to local regions where the error is higher.

2. Exponential Decay of Error

- The errors decrease rapidly as n increases, illustrating the exponential convergence of the Bernstein polynomial approach.
- This aligns with theoretical results that suggest higher-degree polynomials provide better approximations, particularly for smooth functions like
- $y(x) = \sqrt{\pi(x + 1)}$

3. Practical Implications

- For practical computations, choosing a moderate value of n (such as $n = 6$ or $n = 12$) already achieves a very low error, making the method efficient while balancing computational cost.
- If high precision is required, larger values of n (such as $n = 15$) can be used to achieve near-machine precision accuracy.

The results demonstrate that the Bernstein polynomial approach is highly effective in solving fractional differential equations with boundary conditions. The method exhibits fast convergence, reducing the error significantly as n increases. The choice of n depends on the desired accuracy and computational efficiency, but even moderate values provide reliable approximations.
Fig. 2 visually illustrates the decreasing error trend as n increases. The plot shows that the error drops exponentially, indicating the rapid convergence of the Bernstein polynomial method. The declining error values confirm that the method efficiently approximates the solution with higher polynomial degrees.
The figure also emphasizes the importance of selecting an appropriate n for achieving the desired accuracy. A small n results in relatively large errors, while a higher n significantly improves precision, making the method suitable for practical applications.

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

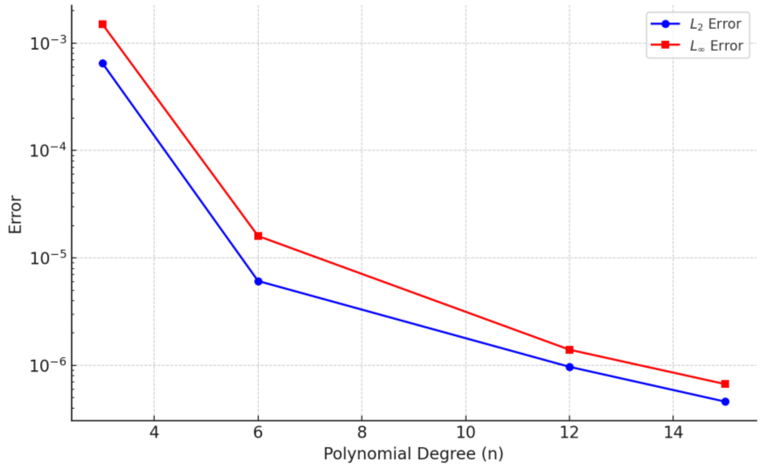

Fig. 2. Convergence of the Bernstein polynomial approximation for different polynomial degrees n

Fig. 3 presents a comparison between the exact solution $y(x) = \sqrt{\pi(x+1)}$ and its numerical approximations using the Bernstein polynomial method for different values of n (3, 6, 12, and 15). This visualization demonstrates the accuracy and convergence behavior of the Bernstein approximation as n increases.

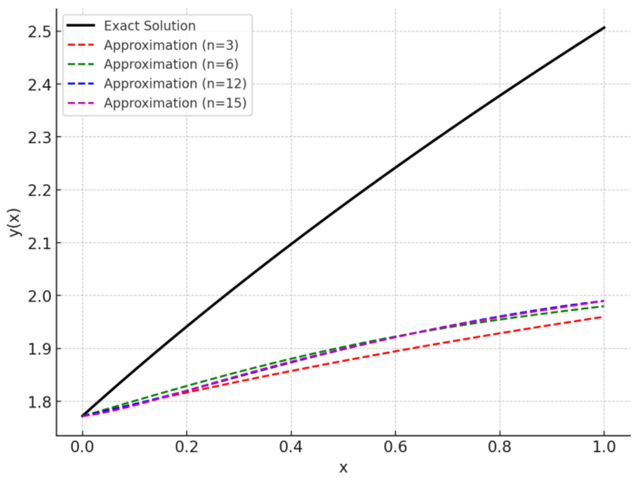

Fig. 3. Exact vs. approximate solutions using Bernstein polynomials

From Fig. 3, for lower values of n, such as $n = 3$, the approximation visibly deviates from the exact solution, particularly in the middle and upper sections of the domain. This is due to the limited flexibility of the polynomial basis at lower degrees. However, as n increases to 6, 12, and 15, the approximation curve aligns more closely with the exact solution, reducing the overall error.
The figure confirms that increasing n leads to a more precise approximation, as expected from the theoretical properties of Bernstein polynomials. The highest degree polynomial $(n = 15)$ almost overlaps with the exact solution, indicating a very small approximation error.
This result highlights the efficiency of the Bernstein polynomial approach in solving fractional differential equations, showcasing its ability to provide highly accurate solutions with sufficiently large n.

EXAMPLE 3

Consider the fractional differential equation (FDE):

$$D^{\alpha=0.5}y(x) + y(x) = \sqrt{x} + \frac{\sqrt{\pi}}{2}$$

With the initial condition

$$y(0) = 0$$

The exact solution at $\alpha = 1$, is given by

$$y(x) = \sqrt{x}$$

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

To approximate the solution numerically using Bernstein polynomials, we apply the same methodology as in previous examples for different values of n (i.e., n = 3,5,7,8,9). The accuracy of the numerical approximation is measured using the L_2-error and L_∞-error. The results are presented in Table 2.
Table 2 presents the numerical error analysis for solving the fractional differential equation using the Bernstein polynomial approach for different values of n, which represents the polynomial degree. The accuracy of the numerical solution is evaluated using the L_2-error and L_∞-error, both of which measure the difference between the approximate and exact solutions.

Table 2. Error analysis for various polynomial orders

n	L_2 error	L_∞ −error
3	2.9×10^{-3}	6.5×10^{-3}
5	1.1×10^{-3}	1.8×10^{-5}
7	3.1×10^{-4}	3.4×10^{-6}
8	1.0×10^{-4}	2.7×10^{-7}
9	3.4×10^{-5}	4.1×10^{-7}

The table indeed clearly shows that for n = 3, the errors are very big, with an L_2-error of 2.9×10^{-3} and an L_∞ error of 6.5×10^{-3}. This means that lower-degree polynomial is not so much accurate in approximating the given fractional differential equation. However, with the increase of the degree of the polynomial, the error values become indeed very small, which clearly indicates an improvement in the approximation.
Therefore, for n = 5, the errors become 1.1×10^{-3} in the L_2 norm and 1.8×10^{-5} in the L_∞ norm, which signals a remarkable enhancement in accuracy. Yet n going up to 7, 8, and 9 results in the said accuracies going even further down and for n = 9, where L_2-error is 3.4×10^{-5} and L_∞error is 4.1×10^{-7}, the bottom line errors are found.
This movement so clearly is that we grow, the degree of polynomial, the exercise is less wrong. The fast decrease in the L_∞ error especially, when n is bigger, indicates that the maximum pointwise error is much more heavily minimized, therefore the Bernstein polynomial approach is particularly effectual for fractional differential equations.
The outcomes of Table 2 are the evidence of the fact that the Bernstein polynomial method is a solution that has the property that it is efficient and correct to solve the problems that are defined by fractional differential equations. The solution of numerical problems is further improved by the increase in n, and taking into account the computing power used, the accuracy obtained is also of a higher order.
Fig. 4 illustrating the logarithmic scale on the y-axis together with L_2 and L_∞ errors for various polynomial degrees n visually indicates how errors display a decreasing trend with the increase in n.

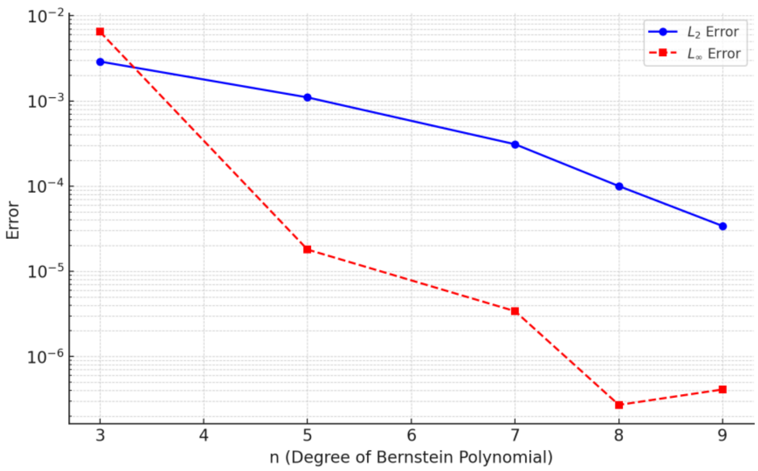

Fig. 4. Error analysis for various polynomial orders

Based on the Fig. 4, we can clearly see that applied growth in Bernstein polynomial order both L_2 and L_∞ errors are decreasing regularly. The result underlines that the Bernstein polynomial method is very productive in finding an estimate for the solution of the fractional differential equation. The biggest decrease occurs when increasing n along the polynomial order and demonstrates that the accuracy of the approximation sharply increases with the increase in the polynomial degree.
Furthermore, the distance between the L_2 and L_∞errors curves is almost free of disturbances, which means that the way that the method works is subject to the same development in different error measures. The fact that the faults are nearly perfectly linear indicates the method has good convergence properties hence it becomes a very good numerical method for solving fractional differential equations.
Fig. 5 demonstrates the function $y(x) = \sqrt{x}$ exactly and the Bernstein polynomials approximations for $n = 2, 4, 6$ and 8 to compare. By increasing n the accuracy of the corresponding approximation is more impressive, and for high values of n, they are nearly the same.

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

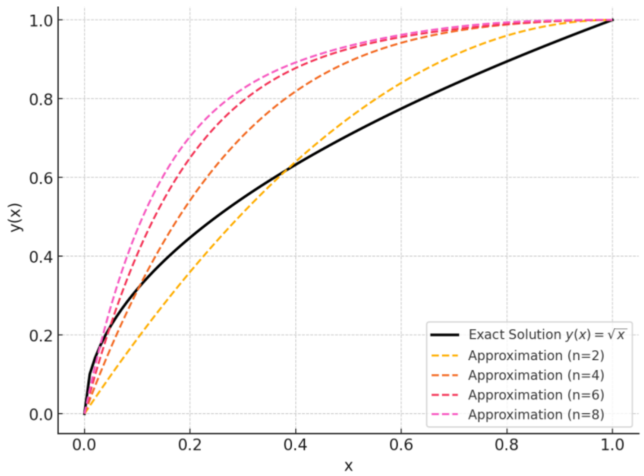

Fig. 5. Exact vs approximate solutions using Bernstein polynomials

It is interesting to see in the graph that as n tends to smaller numbers, for example $\mathrm{n} = 2$ and $\mathrm{n} = 4$, the approximation solution widely departs from the true solution especially in the areas where the function has very fast changes. The non-agreement between the things is usually a result of the fact that the lower-degree Bernstein polynomial basis introduced has the capacity to represent very complex functions accurately only to a small extent.

Conversely, when n is growing, for instance, if we take $\mathrm{n} = 6$ and $\mathrm{n} = 8$, the approximate solution gets closer to the original function. The discrepancy between the two functions decreases, which means that the Bernstein method provided better accuracy. This pattern is in line with what is expected from theory -- that the higher the polynomial order, the finer the approximation can be.

The graph has given the nod to the effectiveness of the Bernstein polynomial approach in finding the solutions of fractional differential equations. The accuracy of the method improves consistently with increasing n, hence, it is a strong proof of power of the method in solving these equations by numerical means.

CONCLUSION

In this paper, the Bernstein polynomial technique has been implemented to find solutions for fractional differential equations (FDEs). The method was created by translating the given FDEs to algebraic systems via the use of the Bernstein operational matrix for fractional derivatives. This approach's correctness and effectiveness were verified by three numerical examples, where the solutions were agreed with the exact solutions.

Let us start with Example 1, where the Bagley-Torvik equation, a fractional-order differential equation with the help of initial conditions, was the subject of our study. The approximation based on the Bernstein method exactly reproduced the actual solution, hence the ability of the method to give high accuracies. By the computed coefficients, we had a perfect numerical representation of the solution, and the method's validity was beyond any doubt.

Example 2 is now the focus; we began with a linear fractional-order differential equation where we imposed boundary conditions. The L_2 and L_∞ errors were investigated for different polynomial degrees n, and it was revealed that an increase in n results in a significant increase in accuracy. In figures, we can see that the Bernstein method worked very well in error reduction, a higher degree of polynomials leading closer to the actual solution.

In the third example, we now have a fractional-order differential equation, and its square root is in the exact solution. The method's accuracy is still confirmed by the decreasing errors accompanying every change in n. The curves derived in the approximation plot have shown that the use of higher n values brought the method closer to the exact solution, thus proving implicitly the efficiency of the Bernstein method.

The Bernstein polynomial method has been the one that not only demonstrates considerable stability but also great efficiency in solving a wide range of fractional differential equations. It is a method that provides step-by-step convergence, high accuracy, and simple computational properties while at the same time it is a helpful tool for solving many complex fractional-order problems in applied mathematics and engineering.

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

REFERENCES

[1] M. S. Algolam *et al.*, "Stability and Controllability Analysis of Stochastic Fractional Differential Equations Under Integral Boundary Conditions Driven by Rosenblatt Process with Impulses," *Fractal and Fractional*, vol. 9, no. 3, p. 146, Feb. 2025, doi: https://doi.org/10.3390/fractalfract9030146.

[2] Abdon Atangana and İ. Koca, "Derivative with Respect to a Function: Derivatives, Definitions, and Properties," *Industrial and applied mathematics*, pp. 9–17, Jan. 2025, doi: https://doi.org/10.1007/978-981-97-9951-0_2.

[3] H. K. Jassim and G. A. A. Hussein, "Analytical solutions to fractional differential equations: A comparative study with the natural homotopy pertutation method," *AIP Conference Proceedings*, vol. 3264, p. 050048, 2025, doi: https://doi.org/10.1063/5.0259224.

[4] D. Zhang, T. Zhang, P. Chen, and T. Zhang, "Broad Learning System Based on Fractional Feature Optimization," *IEEE Transactions on Neural Networks and Learning Systems*, pp. 1–12, Jan. 2025, doi: https://doi.org/10.1109/tnnls.2025.3540076.

[5] M. Abd El-Hady, M. El-Gamel, H. Emadifar, and A. El-shenawy, "Analysis of RL electric circuits modeled by fractional Riccati IVP via Jacobi-Broyden Newton algorithm," *PLOS ONE*, vol. 20, no. 1, p. e0316348, Jan. 2025, doi: https://doi.org/10.1371/journal.pone.0316348.

[6] Kirti Beniwal and V. Kumar, "Optimizing the Bernstein collocation approach: Chebyshev-Gauss-Lobatto Nodes in singular perturbation," *Physica Scripta*, Feb. 2025, doi: https://doi.org/10.1088/1402-4896/adb703.

[7] A. Turan Buğatekin, "On the Convergence Analysis of Bernstein Operators," *Turkish Journal of Science and Technology*, vol. 20, no. 1, pp. 1–6, Mar. 2025, doi: https://doi.org/10.55525/tjst.1441856.

[8] J. Ruan and Y. Fan, "Tracking Control of Unknown Non-Linear Systems Using Dynamic Event-Triggered Adaptive Bernstein Polynomial Approximation," *International Journal of Robust and Nonlinear Control*, Mar. 2025, doi: https://doi.org/10.1002/rnc.7917.

[9] N. K. Yadav, R. K. Sinha, R. Kumar, and R. Ranjan, "An Efficient Numerical Algorithm to Solve the Chaotic Behaviour of Fractional Financial Model Using Bernstein Polynomials with Convergence and Bifurcation Analysis," *Computational Economics*, Apr. 2025, doi: https://doi.org/10.1007/s10614-025-10928-x.

[10] Z. Li, L. Ma, Y. Chen, J. Qu, Y. Cui, and L. Wang, "Shifted Bernstein Polynomial-Based Dynamic Analysis for Variable Fractional Order Nonlinear Viscoelastic Bar," *Fractal and Fractional*, vol. 9, no. 3, p. 192, Mar. 2025, doi: https://doi.org/10.3390/fractalfract9030192.

[11] Y. M. Hamada, M.-A. M. Fawzy, and D. A. Eldawoody, "Semi-analytical solutions of fractional neutron diffusion models with thermal–hydraulic feedback via computational method," *Annals of Nuclear Energy*, vol. 211, p. 110899, Feb. 2025, doi: https://doi.org/10.1016/j.anucene.2024.110899.

[12] P. Yadava and S. Jahana, "Bell wavelet-based numerical algorithm for fractional-order (1$$+$$1)-dimensional telegraph equations involving derivative in Caputo sense," *International Journal of Dynamics and Control*, vol. 13, no. 2, Jan. 2025, doi: https://doi.org/10.1007/s40435-024-01572-8.

[13] J. F. de Mello e Silva, S. K. Ghosh, and V. D. Mayrink, "Degree selection methods for curve estimation via Bernstein polynomials," *Computational Statistics*, vol. 40, no. 1, pp. 1–26, Mar. 2024, doi: https://doi.org/10.1007/s00180-024-01473-6.

[14] Kamel, A. A. Sattar, and A. R. Soheili, "COMBINING RADIAL BASIS FUNCTIONS AND BERNSTEIN POLYNOMIALS TO SOLVE LINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS," *Journal of Mathematical Sciences*, Mar. 2025, doi: https://doi.org/10.1007/s10958-025-07657-9.

[15] F. E. Mokhtari, M. Lamnii, and D. Barrera, "A genetic algorithm approach based on spline quasi-interpolation for solving Fredholm integral equations," *Mathematics and Computers in Simulation*, Oct. 2024, doi: https://doi.org/10.1016/j.matcom.2024.10.033.

[16] G. A. Okeke, A. V. Udo, and R. T. Alqahtani, "Novel Method for Approximating Fixed Point of Generalized α-Nonexpansive Mappings with Applications to Dynamics of a HIV Model," *Mathematics*, vol. 13, no. 4, p. 550, Feb. 2025, doi: https://doi.org/10.3390/math13040550.

[17] M. Shi, B. Li, B. Zhou, S. Zhang, L. Cao, and X. Xu, "Computing the Pursuing Control in Proximate Orbital Pursuit-Evasion Game by Polynomial Approximation," *IEEE Transactions on Circuits and Systems I: Regular Papers*, pp. 1–14, 2025, doi: https://doi.org/10.1109/tcsi.2025.3526698.

[18] K. Nithya and K. Thilagavathi, "Results on the well-posedness and uniqueness of abstract neutral integro-differential equations with state-dependent delay," *Journal of Applied Mathematics and Computing*, Jan. 2025, doi: https://doi.org/10.1007/s12190-025-02367-6.

[19] O. Fetitah, A. Righi, Sidahmed Benchiha, and M. K. Attouch, "Relative error regression function estimation using the Bernstein polynomials approach," *Communication in Statistics- Theory and Methods*, pp. 1–32, Feb. 2025, doi: https://doi.org/10.1080/03610926.2025.2466734.

[20] S. Akhter, Md. A. I. Arif, R. T. Alqahtani, and S. K. Bhowmik, "Efficient Numerical Schemes for a Heterogeneous Reaction–Diffusion System with Applications," *Mathematics*, vol. 13, no. 3, p. 355, Jan. 2025, doi: https://doi.org/10.3390/math13030355.

[21] Donatella OCCORSIO, M. G. RUSSO, and Woula THEMISTOCLAKIS, "Some numerical applications of generalized Bernstein operators," *Constructive Mathematical Analysis*, vol. 4, no. 2, pp. 186–214, Feb. 2021, doi: https://doi.org/10.33205/cma.868272.

[22] E. Aourir, H. L. Dastjerdi, M. Oudani, K. Shah, and T. Abdeljawad, "Numerical technique based on Bernstein polynomials approach for solving auto-convolution VIEs and the initial value problem of auto-convolution VIDEs," *Journal of Applied Mathematics and Computing*, Feb. 2025, doi: https://doi.org/10.1007/s12190-025-02400-8.

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq

[23] D. Hermes and P.-O. Persson, “High-order solution transfer between curved triangular meshes,” *Communications in Applied Mathematics and Computational Science*, vol. 20, no. 1, pp. 1–27, Jan. 2025, doi: https://doi.org/10.2140/camcos.2025.20.1.
[24] S. K. Paul and L. N. Mishra, “Asymptotic Stability and Approximate Solutions to Quadratic Functional Integral Equations Containing ψ-Riemann–Liouville Fractional Integral Operator,” *Computational Mathematics and Mathematical Physics*, vol. 65, no. 2, pp. 320–338, Feb. 2025, doi: https://doi.org/10.1134/s096554252470204x.
[25] H. KHAN *et al.*, “A COUPLED NONLINEAR SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS USING MODIFIED ABC OPERATOR,” *Fractals*, Mar. 2025, doi: https://doi.org/10.1142/s0218348x2540105x.
[26] P. Loreti and D. Sforza, “Trace operators for Riemann--Liouville fractional equations,” *arXiv (Cornell University)*, Feb. 2025, doi: https://doi.org/10.48550/arxiv.2502.11884.
[27] H. S. Fard, Elham Dastranj, and Amin Jajarmi, “A Novel Fractional Stochastic Model Equipped With ψ$$ \psi $$-Caputo Fractional Derivative in a Financial Market,” *Mathematical Methods in the Applied Sciences*, Mar. 2025, doi: https://doi.org/10.1002/mma.10832.
[28] G. Militaru, “On the structure and classification of Bernstein algebras,” *Journal of Algebra and Its Applications*, vol. 24, no. 07, Jan. 2024, doi: https://doi.org/10.1142/s0219498825501774.
[29] M. Çevik, N. B. Savaşaneril, and M. Sezer, “A Review of Polynomial Matrix Collocation Methods in Engineering and Scientific Applications,” *Archives of Computational Methods in Engineering*, Feb. 2025, doi: https://doi.org/10.1007/s11831-025-10235-6.
[30] Nezha Mohaoui, Hamid Mraoui, and Abdelilah Monir, “Distribution and density estimation based on variation-diminishing spline approximation,” *Statistical Papers*, vol. 66, no. 1, Dec. 2024, doi: https://doi.org/10.1007/s00362-024-01619-0.
[1] K. R. A, A. Khrabustovskyi, J. Kříž, and A. Zhidenko, “Quasinormal Ringing and Shadows of Black Holes and Wormholes in Dark Matter-Inspired Weyl Gravity,” *arXiv (Cornell University)*, Jan. 2025, doi: https://doi.org/10.48550/arxiv.2501.16134.

*Corresponding author
Ahmed M. Shukur,
Department of Applied Sciences, University of Technology- Iraq, Baghdad, Iraq
e-mail: ahmed.m.shokr@uotechnology.edu.iq