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Review Article

Transforms for Differentiation along with Integration in Complex Analysis Using Coefficient-Wise Convolution

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ABSTRACT

In contemporary mathematical analysis, linear and nonlinear mappings play a fundamental role. In univalent mapping theory and related fields of complex analysis, creating an operator through coefficient-wise convolution and then examining its geometric and class-preserving properties is still a current area of study. The historical development of several well-known differentiation and integration convolution transforms developed in the literature is traced in this survey-oriented review, which also emphasizes their structural advantages, differential recurrences, and unifying connections. The exposition is designed to serve as a stand-alone resource for researchers and graduate students starting projects in this field.

KEYWORDS: Gauss's hypergeometric series, univalent classes, holomorphic mappings, integration transforms, coefficient-wise convolution, differentiation transformations

INTRODUCTION

Let $\mathcal{A}$ designate the collection of all mappings ζ holomorphic throughout an open unit disk $\mathbb{E} = \{\varpi \in \mathbb{C}: |\varpi| < 1\}$ that satisfy the normalization conditions $\zeta(0) = 0$ and $\zeta'(0) = 1$. Any member $\zeta \in \mathcal{A}$ accordingly admits a Maclaurin series matching

$$\zeta(\varpi) = \varpi + \sum_{c=2}^{\infty} s_c\, \varpi^c \qquad (1.1)$$

Furthermore, let $\mathcal{S} \subset \mathcal{A}$ denote the subfamily comprising all normalized holomorphic mappings that remain univalent (injective) across $\mathbb{E}$. Given two members $\zeta, \chi \in \mathcal{A}$, where $\zeta(\varpi)$ is expanded via (1.1) and $\chi(\varpi) = \varpi + \sum_{c=2}^{\infty} e_c\, \varpi^c$, their coefficient-wise convolution—widely termed Hadamard multiplication—is written as $(\zeta * \chi)(\varpi)$ and defined term by term through

$$(\zeta * \chi)(\varpi) = \varpi + \sum_{c=2}^{\infty} s_c\, e_c \varpi^c \qquad (1.2)$$

It follows directly from (1.2) that $\mathcal{A}$ stays closed under convolution; specifically, $(\zeta * \chi)(\varpi) \in \mathcal{A}$ whenever ζ and χ both lie in $\mathcal{A}$.

For arbitrary complex parameters $s, e, c \in \mathbb{C}$ subject to $c \neq 0, -1, -2, -3, \ldots$, Gauss's hypergeometric series ${}_2F_1(s, e, c; \varpi)$ is specified via

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$$ {}_2F_1(s,e,c;\varpi)=\sum_{c=0}^{\infty}\frac{(s)_c(e)_c}{(c)_c}\frac{\varpi^c}{c!}=1+\sum_{c=2}^{\infty}\frac{(s)_{c-1}(e)_{c-1}}{(c)_{c-1}(c-1)!}\varpi^{c-1} \qquad (1.3) $$

where the subscripts 2 and 1 flanking F count the numerator and denominator parameters, respectively, entering each series coefficient. Here, $(\xi)_c$ signifies Pochhammer's ascending factorial, defined inductively as

$$ (\xi)_c=\begin{cases}1, & \text{when } c=0,\\ \xi(\xi+1)(\xi+2)\cdots(\xi+c-1), & \text{when } c\neq 0\end{cases} \qquad (1.4) $$

Expressed using Euler's Gamma function, this ascending factorial obeys the relation

$$ (\xi)_c=\frac{\Gamma(\xi+c)}{\Gamma(\xi)} $$

Within $\mathcal{S}$, the symbols $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{K}$ stand for the standard geometric subfamilies consisting of mappings that are close-to-convex, convex, and origin-starlike throughout $\mathbb{E}$, respectively. The subfamilies $\mathcal{S}^*$ and $\mathcal{C}$ are linked through Alexander's classical duality theorem [1, 2]. In this context, Libera [3] introduced an integral transform as well as demonstrated that both $\mathcal{S}^*$ as well as $\mathcal{C}$ remain invariant under this transformation. Subsequently, Bernardi [4] extended Libera's application to a one-parameter family. Further contributions from various other researchers (such as [5–11]) led to the development of versatile operators, revealing new holomorphic classes that possess specific geometric properties

SURVEY OF CONVOLUTION TRANSFORMS

The utility of convolution is illustrated through the four classical transforms $\Gamma_i:\mathcal{A}\to\mathcal{A}$ $(i\in\{0,1,2,3\})$, as demonstrated by Barnard and Kellogg in [12], and defined as follows

$\Gamma_0\zeta(\varpi)=\varpi\zeta'(\varpi)$ (Alexander differentiation transform [1])

$$ \Gamma_1\zeta(\varpi)=\frac{(\varpi\zeta(\varpi))'}{2}=\frac{\zeta(\varpi)+\varpi\zeta'(\varpi)}{2} $$ (Livingston transform [13])

$$ \Gamma_2\zeta(\varpi)=\int_0^{\varpi}\frac{\zeta(\zeta)}{\zeta}\,d\zeta $$ (Alexander integration transform [1, 2])

$$ \Gamma_3\zeta(\varpi)=\frac{2}{\varpi}\int_0^{\varpi}\zeta\,(\zeta)\,d\zeta $$ (Libera integration transform [3]).

The associated kernel functions $h_i\in\mathcal{A}$ are explicitly given by

$$ h_0(\varpi)=\varpi+\sum_{c=2}^{\infty}c\,\varpi^c=\frac{\varpi}{(1-\varpi)^2}, \qquad h_1(\varpi)=\varpi+\sum_{c=2}^{\infty}\frac{c+1}{2}\varpi^c=\frac{\varpi-\frac{1}{2}\varpi^2}{(1-\varpi)^2} $$

$$ h_2(\varpi)=\varpi+\sum_{c=2}^{\infty}\frac{1}{c}\varpi^c=-\log(1-\varpi), \qquad h_3(\varpi)=\varpi+\sum_{c=2}^{\infty}\frac{2}{c+1}\varpi^c=\frac{-2[\varpi+\log(1-\varpi)]}{\varpi} $$

Reference [14] by Ruscheweyh provides comprehensive coverage of convolution and may be consulted.

We present, in chronological order, the following subsections concerning ten significant convolutional transforms.

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BERNARDI'S ONE-PARAMETER INTEGRATION TRANSFORM (1969)

Bernardi [4] formulated a one-parameter averaging transform $M_\eta: \mathcal{A} \to \mathcal{A}$, defined for any complex parameter η satisfying $\mathrm{Re}(\eta) > -1$ by

$$M_\eta \zeta(\varpi) = \frac{1+\eta}{\varpi^\eta} \int_0^\varpi t^{\eta-1} \zeta(t)\, dt, \qquad \mathrm{Re}(\eta) > -1 \qquad (2.1)$$

Clearly, choosing $\eta = 0$ reproduces Alexander's integration transform Γ_2, whereas setting $\eta = 1$ yields Libera's transform Γ_3. Using coefficient-wise convolution, (2.1) takes the equivalent form

$$M_\eta \zeta(\varpi) = \varpi\, {}_2F_1(1, 1+\eta, 2+\eta; \varpi) * \zeta(\varpi) \qquad (2.2)$$

where ${}_2F_1(1, 1+\eta, 2+\eta; \varpi)$ denotes Gauss's hypergeometric series from (1.3). Bernardi [4] demonstrated that $\mathcal{S}^*$ and $\mathcal{C}$ both stay invariant under M_η, meaning $M_\eta(\mathcal{S}^*) \subset \mathcal{S}^*$ and $M_\eta(\mathcal{C}) \subset \mathcal{C}$. Later refinements regarding convexity and starlikeness preservation for M_η appear in [15, 16] along with cited works.

DIFFERENTIATION TRANSFORM OF RUSCHEWEYH (1975)

Employing convolution methods, Ruscheweyh [5] constructed a differentiation transform $J^\eta: \mathcal{A} \to \mathcal{A}$, defined for every real parameter $\eta > -1$ by

$$J^\eta \zeta(\varpi) = \frac{\varpi}{(1-\varpi)^{\eta+1}} * \zeta(\varpi), \qquad \eta \in \mathbb{R},\ \eta > -1 \qquad (2.3)$$

Whenever $\eta = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ takes non-negative integer values, (2.3) simplifies to the explicit derivative formula

$$J^m \zeta(\varpi) = \frac{\varpi\left(\varpi^{m-1}\zeta(\varpi)\right)^{(m)}}{m!} \qquad (2.4)$$

usually referred to as Ruscheweyh's m-th derivative. Specifically, $J^0\zeta(\varpi) = \zeta(\varpi)$ acts identically, while $J^1\zeta(\varpi) = \varpi\zeta'(\varpi) = \Gamma_0$ coincides with Alexander's differentiation transform. Moreover, J^η admits the hypergeometric convolution representation

$$J^\eta \zeta(\varpi) = \varpi\, {}_2F_1(\eta+1, 1, 1; \varpi) * \zeta(\varpi) \qquad (2.5)$$

and fulfills the differential recurrence relation

$$\varpi\left(J^\eta \zeta(\varpi)\right)' = (\eta+1) J^{\eta+1}\zeta(\varpi) - \eta J^\eta \zeta(\varpi)$$

Numerous studies (see, e.g., [17–19]) have utilized J^η to define and investigate diverse subfamilies of holomorphic univalent mappings.

BI-PARAMETRIC TRANSFORM OF CARLSON AND SHAFFER (1984)

Working with coefficient-wise convolution, Carlson together with Shaffer [20] introduced a bi-parametric mapping $\mathrm{H}(s, c): \mathcal{A} \to \mathcal{A}$ specified by

$$\mathrm{H}(s, c)\zeta(\varpi) = \phi(s, c; \varpi) * \zeta(\varpi) \qquad (2.6)$$

where $\phi(s, c; \varpi) \in \mathcal{A}$ designates Euler's incomplete beta series defined for $c \neq 0, -1, -2, \ldots$ via

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$$\phi(s,c;\varpi)=\sum_{c=0}^{\infty}\frac{(s)_c}{(c)_c}\varpi^{c+1}=\varpi+\sum_{c=2}^{\infty}\frac{(s)_{c-1}}{(c)_{c-1}}\varpi^{c}=\varpi+\sum_{c=2}^{\infty}\frac{\Gamma(c)\Gamma(s+c-1)}{\Gamma(s)\Gamma(c+c-1)}\varpi^{c} \qquad (2.7)$$

Comparing (1.3) against (2.7) immediately connects $\phi(s,c;\varpi)$ with Gauss's hypergeometric series through

$$\phi(s,c;\varpi)=\varpi\,{}_2F_1(s,1,c;\varpi) \qquad (2.8)$$

which converts (2.6) into

$$\mathrm{H}(s,c)\zeta(\varpi)=\varpi\,{}_2F_1(s,1,c;\varpi)*\zeta(\varpi) \qquad (2.9)$$

This mapping sends $\mathcal{A}$ onto itself. Provided $s\neq 0,-1,-2,\ldots$, $\mathrm{H}(s,s)$ acts identically on $\mathcal{A}$, while $\mathrm{H}(c,s)$ furnishes a continuous inverse mapping to $\mathrm{H}(s,c)$ whenever $s,c\neq 0,-1,-2,\ldots$. Furthermore, $\mathrm{H}(s,c)$ obeys the differential recurrence identity

$$\varpi\big(\mathrm{H}(s,c)\zeta(\varpi)\big)'=s\mathrm{H}(s+1,c)\zeta(\varpi)-(s-1)\mathrm{H}(s,c)\zeta(\varpi),\qquad c\neq 0,-1,-2,\ldots$$

Particular parameter selections recover earlier transforms: setting $s=\eta+1$ with $c=1$ yields $\mathrm{H}(\eta+1,1)\zeta(\varpi)=J^{\eta}\zeta(\varpi)$, demonstrating that $\mathrm{H}(s,c)$ encompasses (2.3) as a particular instance. Similarly, $\mathrm{H}(2,1)\zeta(\varpi)=\Gamma_0\zeta(\varpi)$ and $\mathrm{H}(3,2)\zeta(\varpi)=\Gamma_1\zeta(\varpi)$. Later, Shanmugam et al. [21] derived sandwich-type subordination theorems governing holomorphic subfamilies associated with $\mathrm{H}(s,c)$.

TRI-PARAMETRIC HYPERGEOMETRIC TRANSFORM OF HOHLOV (1984)

Based upon Gauss's hypergeometric series (1.3), Hohlov [22, 23] formulated a tri-parametric linear convolution transform $H_{s,e,c}:\mathcal{A}\to\mathcal{A}$ given by

$$H_{s,e,c}\zeta(\varpi)=\varpi\,{}_2F_1(s,e,c;\varpi)*\zeta(\varpi) \qquad (2.10)$$

and examined its univalence-preserving and geometric attributes. This family depending on three parameters unifies most classical linear integration and differentiation transforms as special instances. Setting $e=1$ inside (2.10) simplifies $H_{s,1,c}$ directly to Carlson–Shaffer's mapping (2.6); likewise, appropriate parameter choices recover both Ruscheweyh's differentiation transform and Bernardi's integration transform.

Hohlov [23] proved that for every $\varepsilon>0$, the transform $H_{1,e,c+\varepsilon}$ maps $\mathcal{S}$ into itself whenever $e>0$ and $c>c_0(e)+e+2$, where $c_0(e)$ denotes the largest positive real root satisfying the cubic equation

$$y^3-4ey^2-(5e^2+e+1)y-(2e^3+e^2-e)=0$$

Hohlov [23] also established criteria guaranteeing that $H_{s,e,c}$ carries the convex subfamily $\mathcal{C}$ into $\mathcal{S}$. Subsequent invariance behavior exhibited by $H_{s,e,c}$ was analyzed by Mishra with Panigrahi [24].

FRACTIONAL CALCULUS TRANSFORM OF OWA AND SRIVASTAVA (1987)

Within the Riemann–Liouville framework, fractional differentiation having order ξ $(0\leq\xi<1)$ for a holomorphic mapping ζ is defined via

$$J_{\varpi}^{\xi}\zeta(\varpi)=\frac{1}{\Gamma(1-\xi)}\frac{d}{d\varpi}\int_0^{\varpi}\frac{\zeta(\xi)}{(\varpi-\xi)^{\xi}}\,d\xi,\qquad 0\leq\xi<1 \qquad (2.11)$$

with $J_{\varpi}^{0}\zeta(\varpi)=\zeta(\varpi)$, where branch multiplicity inside $(\varpi-\xi)^{-\xi}$ is removed by stipulating $\log(\varpi-\xi)\in\mathbb{R}$ (i.e., $\varpi-\xi>0$). Higher-order fractional derivatives are defined inductively for $\beta\in\mathbb{N}_0$ through

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$$J_{\varpi}^{\beta+\xi}\zeta(\varpi) = \frac{d^{\beta}}{d\varpi^{\beta}} J_{\varpi}^{\xi}\zeta(\varpi), \qquad \beta \in \mathbb{N}_0$$

Drawing upon these concepts, Owa together with Srivastava [25] constructed the transform $\psi^{\xi}: \mathcal{A} \to \mathcal{A}$ for $\xi \geq 0$ ($\xi \neq 2,3,4, \ldots$) via

$$\psi^{\xi}\zeta(\varpi) = \Gamma(2-\xi)\varpi^{\xi} J_{\varpi}^{\xi}\zeta(\varpi), \qquad \xi \neq 2,3,4, \ldots$$

Substituting the Maclaurin expansion (1.1) of $\zeta(\varpi)$ into (2.11) produces the infinite series and beta-convolution expressions

$$\psi^{\xi}\zeta(\varpi) = \varpi + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(2-\xi)}{\Gamma(m+1-\xi)} s_m \varpi^m = \phi(2,2-\xi;\varpi) * \zeta(\varpi) \qquad (2.12)$$

Applying (2.8) to (2.12) yields the equivalent hypergeometric convolution form

$$\psi^{\xi}\zeta(\varpi) = \varpi\, {}_2F_1(2,1,2-\xi;\varpi) * \zeta(\varpi), \qquad \xi \neq 2,3,4, \ldots \qquad (2.13)$$

where $\phi(2,2-\xi;\varpi)$ appears in (2.7). Clearly, $\psi^0\zeta(\varpi) = \zeta(\varpi)$ acts identically, whereas $\psi^1\zeta(\varpi) = \varpi\zeta'(\varpi) = \Gamma_0$ recovers Alexander's differentiation transform. Detailed distortion and characterization theorems governing ψ^{ξ} appear in [9, 26].

INTEGRATION-TYPE TRANSFORM OF NOOR (1999)

Modeled analogously to Ruscheweyh's differentiation transform, K. I. Noor with M. A. Noor [6] and Noor [8] proposed an integration-type convolution mapping as follows. Set $\zeta_c(\varpi) = \frac{\varpi}{(1-\varpi)^{c+1}}$ ($c \in \mathbb{N}_0$), and define $\zeta_c^{(-1)}(\varpi)$ implicitly via the convolution identity

$$\zeta_c(\varpi) * \zeta_c^{(-1)}(\varpi) = \frac{\varpi}{(1-\varpi)^2}$$

Noor's integration transform having order c, denoted $I_c: \mathcal{A} \to \mathcal{A}$, is then specified through

$$I_c\zeta(\varpi) = \zeta_c^{(-1)}(\varpi) * \zeta(\varpi) \qquad (2.14)$$

Direct evaluation confirms that $I_0\zeta(\varpi) = \varpi\zeta'(\varpi) = J^1\zeta(\varpi)$ (Alexander's differentiation transform) and $I_1\zeta(\varpi) = \zeta(\varpi) = J^0\zeta(\varpi)$ (the identity mapping), generalizing Γ_0. This reciprocal duality between (I_0, I_1) and (J^0, J^1) motivates classifying I_c as an integration-type transform. Expressed via Gauss's hypergeometric series, (2.14) becomes

$$I_c\zeta(\varpi) = \varpi\, {}_2F_1(2,1,c+1;\varpi) * \zeta(\varpi) \qquad (2.15)$$

Moreover, (2.14) leads directly to the differential recurrence relation

$$\varpi\big(I_{c+1}\zeta(\varpi)\big)' = (c+1)I_c\zeta(\varpi) - cI_{c+1}\zeta(\varpi)$$

Holomorphic subfamilies defined via I_c remain an active topic of investigation (cf. [7, 10]).

MULTI-PARAMETER TRANSFORM OF DZIOK AND SRIVASTAVA (1999)

Extended hypergeometric series ${}_dF_s(\xi_1, \ldots, \xi_d; \beta_1, \ldots, \beta_s; \varpi)$ generalize (1.3) to d numerator parameters and s denominator parameters via

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$$ {}_dF_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s;\varpi)=\sum_{c=0}^{\infty}\frac{(\xi_1)_c\cdots(\xi_d)_c}{(\beta_1)_c\cdots(\beta_s)_c}\frac{\varpi^c}{c!} $$

where $d,s\in\mathbb{N}_0$ fulfill $d\le s+1$, and $\beta_i\neq 0,-1,-2,\dots$ for every $i=1,2,\dots,s$. Note that $\varpi\,{}_dF_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s;\varpi)\in\mathcal{A}$. Utilizing this extended series, Dziok together with Srivastava [27–29] defined a multi-parameter linear convolution transform ${}_dH_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s):\mathcal{A}\to\mathcal{A}$ by

$$ {}_dH_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)=\varpi\,{}_dF_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s;\varpi)*\zeta(\varpi) \qquad (2.16) $$

for $d\le s+1$ $(d,s\in\mathbb{N}_0)$. This transform obeys a system of contiguous differential recurrence relations:

$$ \begin{aligned}\varpi\big({}_dH_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)\big)' &= \xi_1\,{}_dH_s(\xi_1+1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)-(\xi_1-1)\,{}_dH_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)\\ &= \xi_2\,{}_dH_s(\xi_1,\xi_2+1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)-(\xi_2-1)\,{}_dH_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)\\ &\ \ \vdots\\ &= \xi_d\,{}_dH_s(\xi_1,\dots,\xi_d+1;\beta_1,\dots,\beta_s)\zeta(\varpi)-(\xi_d-1)\,{}_dH_s(\xi_1,\dots,\xi_d;\beta_1,\dots,\beta_s)\zeta(\varpi)\end{aligned} $$

Selecting appropriate indices d,s with parameter tuples $(\xi_1,\dots,\xi_d)$ and $(\beta_1,\dots,\beta_s)$ recovers the transforms due to Hohlov, Carlson–Shaffer, Ruscheweyh, Bernardi, Owa–Srivastava, Noor, as well as Choi–Saigo–Srivastava. Differential subordination and superordination theorems involving ${}_dH_s$ were established in [30, 31].

GENERALIZED INTEGRATION TRANSFORM OF SAIGO, CHOI, AND SRIVASTAVA (2002)

A bi-parametric extension of Noor's integration transform was formulated by Saigo, Choi, together with Srivastava [32] as the convolution mapping $I_{\eta,\kappa}:\mathcal{A}\to\mathcal{A}$, defined by

$$ I_{\eta,\kappa}\zeta(\varpi)=\big(\zeta_{\eta,\kappa}*\zeta\big)(\varpi),\qquad \eta>-1,\ \kappa>0 \qquad (2.17) $$

where the kernel function $\zeta_{\eta,\kappa}(\varpi)$ is determined through the convolution condition

$$ \frac{\varpi}{(1-\varpi)^{\eta+1}}*\zeta_{\eta,\kappa}(\varpi)=\frac{\varpi}{(1-\varpi)^{\kappa}} $$

Using Gauss's hypergeometric series, (2.17) takes the explicit form

$$ I_{\eta,\kappa}\zeta(\varpi)=\varpi\,{}_2F_1(\kappa,1,\eta+1;\varpi)*\zeta(\varpi) \qquad (2.18) $$

Evidently, choosing $\kappa=2$ simplifies $I_{\eta,2}$ to Noor's transform (2.14), while setting $\kappa=2$ with $\eta=1-\xi$ yields Owa–Srivastava's fractional transform (2.12). Furthermore, $I_{0,2}\zeta(\varpi)=\varpi\zeta'(\varpi)$ and $I_{1,2}\zeta(\varpi)=\zeta(\varpi)$, justifying its integration-type designation. Directly from (2.17), one deduces two recurrence identities:

$$ \varpi\Big(I_{\eta+1,\kappa}\zeta(\varpi)\Big)'=(\eta+1)I_{\eta,\kappa}\zeta(\varpi)-\eta I_{\eta+1,\kappa}\zeta(\varpi) $$

$$ \varpi\Big(I_{\eta,\kappa}\zeta(\varpi)\Big)'=\kappa I_{\eta,\kappa+1}\zeta(\varpi)-(\kappa-1)I_{\eta,\kappa}\zeta(\varpi) $$

Further geometric characterizations and inclusion properties connected with $I_{\eta,\kappa}$ have been investigated in [33, 34].

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ZETA-FUNCTION TRANSFORM OF SRIVASTAVA AND ATTIYA (2007)

Recall that Hurwitz–Lerch's generalized zeta series $\varphi(\kappa, e; \varpi)$ is defined via

$$\varphi(\kappa, e; \varpi) = \sum_{c=0}^{\infty} \frac{\varpi^c}{(c+e)^{\kappa}} = e^{-\kappa} + \frac{\varpi}{(1+e)^{\kappa}} + \sum_{c=2}^{\infty} \frac{\varpi^c}{(c+e)^{\kappa}} \qquad (2.19)$$

where $e \in \mathbb{C}\backslash\{0, -1, -2, -3, \dots\}$, $\kappa \in \mathbb{C}$ satisfies $\mathrm{Re}(\kappa) > 1$, and $|\varpi| < 1$. Forming the coefficient-wise convolution of a normalized Hurwitz–Lerch zeta kernel with members of $\mathcal{A}$, Srivastava together with Attiya [35] constructed the linear family $M_{\kappa,e}: \mathcal{A} \to \mathcal{A}$ via

$$M_{\kappa,e}\zeta(\varpi) = L_{\kappa,e}(\varpi) * \zeta(\varpi) \qquad (2.20)$$

where $L_{\kappa,e} \in \mathcal{A}$ takes the form

$$L_{\kappa,e}(\varpi) = (1+e)^{\kappa}[\varphi(\kappa, e; \varpi) - e^{-\kappa}] = \varpi + \sum_{c=2}^{\infty} \left(\frac{1+e}{c+e}\right)^{\kappa} \varpi^c \qquad (2.21)$$

Notice that $M_{0,e}$ acts identically on $\mathcal{A}$, whereas $M_{-\kappa,e}$ operates inversely to $M_{\kappa,e}$. Combining (2.20) and (2.21) yields the recurrence relation

$$\varpi\left(M_{\kappa,e}\zeta(\varpi)\right)' = (1+e)M_{\kappa-1,e}\zeta(\varpi) - eM_{\kappa,e}\zeta(\varpi)$$

Suitable parameter selections for κ and e recover numerous classical differentiation and integration transforms (see [35] for a thorough classification). Subordination properties were derived in [35], while applications involving strongly convex and strongly starlike classes were studied in [36].

ITERATIVE FRACTIONAL TRANSFORM OF AL-OBOUDI AND AL-AMOUDI (2008)

Generalizing Owa–Srivastava's fractional transform [25], Al-Oboudi together with Al-Amoudi [37] introduced an iterative fractional-differential transform $J_{\eta}^{c,\xi}: \mathcal{A} \to \mathcal{A}$. For parameters $\eta \geq 0$, $c \in \mathbb{N}_0$, and $\xi \geq 0$ ($\xi \neq 2,3,4, \dots$), this mapping is defined inductively by

$$\begin{aligned}
J_{\eta}^{0,\xi}\zeta(\varpi) &= \zeta(\varpi), \\
J_{\eta}^{1,\xi}\zeta(\varpi) &= (1-\eta)\psi^{\xi}\zeta(\varpi) + \eta\varpi\left(\psi^{\xi}\zeta(\varpi)\right)' := J_{\eta}^{\xi}\zeta(\varpi), \qquad \xi \neq 2,3,4, \dots, \\
J_{\eta}^{2,\xi}\zeta(\varpi) &= J_{\eta}^{\xi}\left(J_{\eta}^{1,\xi}\zeta(\varpi)\right), \\
&\vdots \\
J_{\eta}^{c,\xi}\zeta(\varpi) &= J_{\eta}^{\xi}\left(J_{\eta}^{c-1,\xi}\zeta(\varpi)\right), \qquad c \in \mathbb{N}.
\end{aligned}$$

Using the Maclaurin expansion (1.1) of $\zeta(\varpi)$, this transform admits the series expansion

$$J_{\eta}^{c,\xi}\zeta(\varpi) = \varpi + \sum_{m=2}^{\infty} Z_{m,c}(\xi, \eta) s_m \varpi^m, \qquad c \in \mathbb{N}_0,\ \xi \neq 2,3,4, \dots \qquad (2.22)$$

where the multiplier coefficient $Z_{m,c}(\xi, \eta)$ takes the form

$$Z_{m,c}(\xi, \eta) = \left[\frac{\Gamma(m+1)\Gamma(2-\xi)}{\Gamma(m+1-\xi)}(1+\eta(m-1))\right]^c \qquad (2.23)$$

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Equivalently, (2.22) admits representation as an c-fold convolution involving Euler's incomplete beta series $\phi(2,2-\xi;\varpi)$:

$$J_{\eta}^{c,\xi}\zeta(\varpi) = \phi(2,2-\xi;\varpi) * \chi_{\eta}(\varpi) * \cdots * \phi(2,2-\xi;\varpi) * \chi_{\eta}(\varpi) * \zeta(\varpi) \qquad (2.24)$$

or, via (2.8), in Gauss hypergeometric convolution form as

$$J_{\eta}^{c,\xi}\zeta(\varpi) = \varpi\,{}_2F_1(2,1,2-\xi;\varpi) * \chi_{\eta}(\varpi) * \cdots * \varpi\,{}_2F_1(2,1,2-\xi;\varpi) * \chi_{\eta}(\varpi) * \zeta(\varpi) \qquad (2.25)$$

where the auxiliary kernel $\chi_{\eta}(\varpi)$ is specified by

$$\chi_{\eta}(\varpi) = \frac{\varpi - (1-\eta)\varpi^2}{(1-\varpi)^2}$$

When $c = 1$ and $\eta = 0$, $\chi_0(\varpi) = \frac{\varpi}{1-\varpi}$ serves as a convolution unit in $\mathcal{A}$, reducing $J_0^{1,\xi}$ directly to Owa–Srivastava's transform (2.12).

CONCLUDING SUMMARY

Hadamard convolution together with linear transforms constructed via convolution remain indispensable instruments across contemporary complex analysis. Representing integration and differentiation operators within a unified convolution framework provides an effective algebraic mechanism for investigating class invariance, inverse structures, and geometric mapping behavior. As highlighted throughout this survey, nearly all prominent convolution transforms in univalent mapping theory are hypergeometric in structure: almost every one emerges as a particular parameter case of Hohlov's tri-parametric transform, which itself is encompassed within the general Dziok–Srivastava family.

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